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ON SOME CATEGORY OF CHU SPACES OVER CATEGORY OF S-ACTS

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Abstract. This work continues the earlier works of the same authors. We consider a category $Chu(S-Act)$ of Chu spaces over the category of S-acts where S is commutative monoid. Structure of a monoidal category necessary for the determining the Chu spaces is given by the tensor product of S-acts. An important characteristic of the category is the ability to move to the limits and colimits. It is known that for the existence of (co)limits it is necessary and sufficient the existence of (co)equalizers and (co)products. (Co)limits are computed through (co)product and (co)equalizers in a standard way. In category $Chu(S-Act)$ coequalizers and coproducts exist. There is the description of Chu transforms for which equalizer exists in category $Chu(S-Act)$. In this work we give necessary condition for existence of product of Chu spaces in the category $Chu(S-Act)$.

Keywords: Chu spaces, Chu construction, S-Act, monoidal category, product, coproduct.

The construction of Chu that later led to the notion of a Chu space appeared in Po Hsiang-Chu's Master's thesis on category theory and was published in 1979 [1]. It allows for a given symmetric monoidal closed category and fixid object to build a new category. If the original category was $*$-autonomous, then in the resulting category the selected object becomes dual. In particular, the category $Chu(\mathcal{V})$ is built Chu objects, or Chu spaces, where $\mathcal{V}$ is a category [2]. It is done this way. Let $\otimes: \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ is bifunctor, that is the tensor product on the category $\mathcal{V}$ and D is a fixed object of $\mathcal{V}$. An arrow $r: A \otimes X \to D$ is called Chu spaces of $\mathcal{V}$, where $A, X \in Ob(\mathcal{V})$. An arrow $(f, g): (A_1 \otimes X_1 \to D) \to (A_2 \otimes X_2 \to D)$ of $Chu(\mathcal{V})$ consists of a twice of arrows $f: A_1 \to A_2$, $g: X_2 \to X_1$ of $\mathcal{V}$ such that $r_1 \circ (1_{A_1} \otimes g) = r_2 \circ (f \otimes 1_{X_2})$ is called transform of Chu spaces or simply Chu transform.

The categories of Chu spaces are actively studied and studied in connection with by interplication in terms of Chu spaces objects of topology, algebra, computer science, game theory [6, 7. For the case when $\mathcal{V}$ is fixed category general properties of the Chu space category studied in the work [2].

In [8], categories of Chu spaces over the category of S-acts where S is a commutative monoid are studied. Structure of a monoidal category necessary for the determining the Chu spaces is given by the tensor product of S-acts. As a rule the category of Chu spaces over a monoidal category inherits many of its properties, including completeness, that is the existence of limits and colimits. this applies to the category $Chu(S-Act, D)$. However, this is not so in the category $Chu(S-Act)$ of all Chu spaces, that is, when D is not fixed (the same category was first introduced in [8]). This category is complete, but there are examples showing that limits do not always exist [8].

An important characteristic of the category is the ability to move to the limits and colimits. It is known that for the existence of (co)limits it is necessary and sufficient the existence of (co)equalizers and (co)products. (Co)limits are computed through (co)product and (co)equalizers in a standard way. In [8] questions associated with equalizers and co-equalizers for $Chu(S-Act)$ are studied. In [8] there is the description of Chu transforms for which equalizer exists in category $Chu(S-Act)$. In this work we give necessary condition for existence of product of Chu spaces in the category $Chu(S-Act)$. This result is new.

Suppose that $\mathcal{V}$ is a symmetric monoidal closed category with a tensor product $\otimes$.

We define a category $Chu(\mathcal{V})$ [8] that has as objects $r: A \otimes X \to D$ is an arrow of $\mathcal{V}$ where $A, X, D \in Ob(\mathcal{V})$. Further these arrows will be call Chu spaces. An arrow $(f, g, h): (A_1 \otimes X_1 \to D_1) \to (A_2 \otimes X_2 \to D_2)$ of $Chu(\mathcal{V})$ consists of a triple of arrows $f: A_1 \to A_2$, $g: X_2 \to X_1$, $h: D_1 \to D_2$ of $\mathcal{V}$ (note the direction of the second arrow) such that $h \circ r_1 \circ (1_{A_1} \otimes g) = r_2 \circ (f \otimes 1_{X_2})$. Further arrows of Chu spaces will be call transform of Chu spaces or simply Chu transform.

Recall some concepts from the category theory [5] and S-acts theory [3, 4]. Let $\mathcal{V}$ is the category.

The equalizer of two morphisms f and g between two objects A and B is the monomorphism $e: E \to A$ such that $f \circ e = g \circ e$ and if $h: F \to A$ is a morphism of $\mathcal{V}$ such that $f \circ h = g \circ h$ then there is a unique morphism $u: F \to E$ such that $h = e \circ u$.

The coequalizer of two morphisms f and g between two objects A and B is the epimorphism $q: B \to Q$ such that $q \circ f = q \circ g$ and if $h: B \to F$ is a morphism of $\mathcal{V}$ such that $h \circ f = h \circ g$ then there is a unique morphism $v: Q \to F$ such that $h = v \circ q$.

The product of objects X_j $(j \in J)$ is an object X (often denoted $\prod_{j \in J} X_j$) together with a morphisms $\pi_j: X \to X_j$ $(j \in J)$ such that for every object Y and morphisms $f_j: Y \to X_j$ $(j \in J)$ there exists a unique morphism $f: Y \to X$ such that $\pi_j \circ f = f_j$ for all $j \in J$.

The coproduct of objects X_j $(j \in J)$ is an object (often denoted $\coprod_{j \in J} X_j$) together with a morphisms $i_j: X_j \to X$ $(j \in J)$ that satisfy the following universal property: for every object Y and morphisms $f_i: X_j \to Y$ $(j \in J)$ there exists a unique morphism $f: X \to Y$ such that $f \circ i_j = f_j$ for all $j \in J$.

Throughout, by S we denote a monoid and by 1 we denote unity in S. An algebraic system $\langle A; s \rangle_{s \in S}$ of the language $L_S = \{s | s \in S\}$ is a (left) S–act if $s_1(s_2 a) = (s_1 s_2)a$ and $1a = a$ for all $s_1, s_2 \in S$ and $a \in A$, that is S-act is a set A on which S acts in the left and the unity acts identically. We denote the class of all S–acts by $S - Act$. The class of all S–acts forms a category where objects are S-acts and morphisms are homomorphisms of S-acts.

Note that the product of S–acts A_j $(j \in J)$ is S–act $\prod_{j \in J} A_i$ where $(sa)(j) = (sa(j))$ for all $j \in J$, $a \in A$, $s \in S$; the coproduct of S–acts A_j $(j \in J)$ is a disjunctive union of this S–acts.

Suppose that S is a commutative monoid. The tensor product $A \otimes B$ of S-acts A и B is defined as a factor act of S-act $A \times B$ by congruence generated by a set $\{(sa, b), (a, sb) | a \in A, b \in B, s \in S\}$. The class of the congruence with representative (a, b) will be denote by $a \otimes b$ where $a \in A$ и $b \in B$. Note that the action of the monoid S on the set $A \otimes B$ has the following property: for any $a \in A, b \in B, s \in S$

$$s(a \otimes b) = sa \otimes b = a \otimes sb.$$

Theorem 1. [8] *Let S is a commutative monoid. Then $S - Act$ is a symmetric monoidal closed category with a tensor product $\otimes$.*

In [8], the Chu construction is applied to the category $S - Act$, more exactly, two categories $Chu(S - Act)$ and $Chu(S - Act, D)$ are introduced, where S is a commutative monoid.

In the category $Chu(S - Act, D)$ for any Chu spaces and Chu transforms there are equalizers, coequalizers, products and coproducts [8]. So for this category there are limits and colimits.

In the category $Chu(S - Act)$ there are similar results for coequalizers and coproducts [8]:

Theorem 2. [8] *Let $r: A \otimes X \to D_1$, $s: B \otimes Y \to D_2$ are the Chu spaces and (f_1, g_1, h_1), $(f_2, g_2, h_2): r \to s$ are the Chu transforms of $Chu(S - Act)$. Then the coequalizer of two Chu transforms (f_1, g_1, h_1), (f_2, g_2, h_2) is the Chu transform $(f, g, h): s \to t$ where $t: Q \otimes E \to W$, $Q = B/\nu(f_1, f_2)$, $W = D/\nu(h_1, h_2)$, $E = \{y \in Y | g_1(y) = g_2(y)\}$, f, h are canonical epimorphisms, g is a natural embedding.*

Theorem 3. [8] *Let* $r_i : A_i \otimes X_i \to D_i$ *are the Chu spaces, where* $i \in I$. *Then there is the coproduct* r *of* r_i *(*$i \in I$*), where* $r: \coprod_{i \in I} A_i \otimes \prod_{i \in I} X_i \to \coprod_{i \in I} D_i$ *such that* $r(a \otimes x) = r_i(a \otimes x(i))$ *for any* $i \in I$, $a \in A_i$, $x \in \prod_{i \in I} X_i$.

Let's look at the issues related to equalizers and products in the category $Chu(S - Act)$. The follow theorem gives the description of Chu transforms for which equalizer exists.

Theorem 4. [9] *Let* $r: A \otimes X \to D_1$, $s: B \otimes Y \to D_2$ *are the Chu spaces and* $(f_1, g_1, h_1), (f_2, g_2, h_2): r \to s$ *are the Chu transforms. Then the equalizer of two Chu transforms* $(f_1, g_1, h_1), (f_2, g_2, h_2)$ *exists if and only if there is an element* $a \in A$ *such that*

1) $f_1(a) = f_2(a)$;
2) $r(a \otimes g_1(y)) = r(a \otimes g_2(y))$ for any $y \in Y$;
3) $h_1 \circ r(a \otimes x) = h_2 \circ r(a \otimes x)$ for any $x \in X$.

It gives necessary condition for existence of product of Chu spaces in the category $Chu(S - Act)$.

Theorem 5. *Let* $r_1: A_1 \otimes X_1 \to D_1$, $r_2: A_2 \otimes X_2 \to D_2$ *are the Chu spaces. Suppose that Chu space* $r: A \otimes X \to D$ *with Chu transforms* $(f_1, g_1, h_1): r \to r_1$, $(f_2, g_2, h_2): r \to r_2$ *are the product of* r_1 *and* r_2. *Then* $A = f_1(A_1) \times f_2(A_2)$, $X = X_1 \coprod X_2$ *and* $D = D_1 \times D_2$.

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