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OPTIMIZATION METHOD OF SOLVING 2D PROBLEMS OF DESIGNING SHIELDING AND CLOAKING DEVICES

Abstract. We study inverse problems for the 2D model of electric conductivity that arise when designing circular shielding or cloaking shells and other functional devices used to control static electric fields. It is assumed that the shells consist of a finite number of circular layers filled with isotropic (inhomogeneous or homogeneous) media. By optimization method our inverse problems are reduced to corresponding control problems in which the role of controls is played by shell layer conductivities. The solvability of the direct and control problems is proved. A numerical algorithm based on the particle swarm optimization method is proposed for solving control problems and some simulation results are discussed.

Key words electric conductivity model, inverse problem, optimization method, cloaking shielding, solvability, particle swarm optimization.

1. Introduction

In recent years new direction in mathematical and technical physics has been intensively developed. It concerns developing methods of solving problems of designing special functional devices for controlling physical fields. An important example of such devices is a material shell in the form of the circular ring (or spherical layer in the space R^3) filled with a anisotropic inhomogeneous medium in general case. This shell can be used, for example, to cloak any object placed inside it from detection using a wave or static physical field.

The first works [1, 2] in this field were devoted to designing invisibility cloaking devices with respect to electromagnetic waves on the basis of the transformation optics method proposed in [1]. Then the main results from electromagnetic cloaking were expanded to acoustic cloaking [3] and later to cloaking static fields (see, e.g. [4–8]). However, we emphasize that the solutions obtained in these papers possess several drawbacks. In particular, it is very difficult to implement these solutions technically. One of approaches of overcoming these difficulties consists of using the optimization method of solving inverse problems. The method to which one should referred to as inverse design method [9] was applied firstly in [10]. Also, it was applied in [11–19] when studying theoretically electromagnetic or acoustic cloaking problems.

The paper consists of two parts. In the first part we formulate firstly the direct and inverse problems for electrical conductivity model that are connected with designing circular shielding or cloaking shells. Then we reduce our inverse problems to respective control problems and present two theorems about global solvability both direct and control problems. In the second part we develop a numerical method of solving our control problems based on particle swarm optimization method [20] using the scheme proposed in [21–22] when solving problems of designing thermal shielding or cloaking devices. Finally, we discuss some results of numerical experiments.

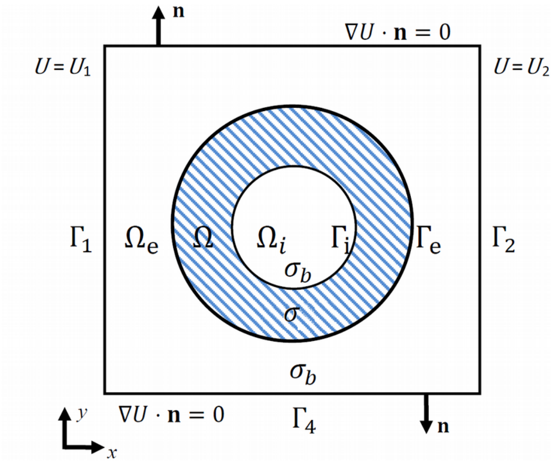
2. Statement of direct and inverse problems

We begin with statement of the general direct conductivity problem considered in a rectangle $D=\{x\equiv(x,y):|x|<x_0,|y|<y_0\}$ with specified numbers $x_0>0$ and $y_0>0$. We assume that an external electric potential U^e is created by two vertical plates $x=\pm x_0$ which are kept at different values U_1 and U_2, while the upper and lower plates $y=\pm y_0$ are electrically insulated. Let us assume also that the medium filling D is homogeneous and isotropic and its electroconductive properties are described by constant conductivity $\sigma_b>0$. If, besides, $U_1=const$, $U_2=const$ then the potential U^e describes one-dimensional (depending on coordinate x) field with constant gradient having the form

$$U^e(x)=\frac{U_2-U_1}{2}\frac{x}{x_0}+\frac{U_1+U_2}{2} \quad . \tag{1}$$

We assume further that there is a material layered shell $\Omega=\{x:a<|x|<b\}$ inside D which consists of M concentric circular layers $\Omega_k=\{r_{k-1}<r=|x|<r_k\},k=1,2,\ldots,M$ where $r_0=a,r_M=b$. Each of these layers is filled with an inhomogeneous in general case isotropic medium, described by variable conductivity $\sigma_k,k=1,2,\ldots,M$.

We assume also that interior $\Omega_i:|x|<a$ and exterior $\Omega_e:|x|>b$ of Ω are filled with the same homogeneous medium having constant electrical conductivity $\sigma_b>0$ (see Fig. 1).



In this case, the direct electrical conductivity problem consists of finding $M+2$ functions, namely, u_i in Ω_i,u_k in $\Omega_k,k=1,2,\ldots,M$ and u_e in Ω_e, which satisfy the equations

$$\sigma_b\Delta u_i=0\in\Omega_i,\sigma_b\Delta u_e=0\in\Omega_e \quad , \tag{2}$$

$$\text{¿}(\sigma_k grad\, u_k)=0\in\Omega_k,k=1,2,\ldots,M \quad , \tag{3}$$

obey the following boundary conditions on the boundary ∂D of D:

$$u_e|_{x=-x_0}=u_1,u_e|_{x=x_0}=u_2,\partial u_e/\partial y|_{y=-y_0}=0 \quad , \tag{4}$$

satisfy the matching conditions on internal Γ_i and external Γ_e components of the boundary Γ of the shell Ω having the form

$$u_i=u,\sigma_b\frac{\partial u_i}{\partial n}=(\sigma\nabla u)\cdot n on \Gamma_i \quad , \quad u_e=u,\sigma_b\frac{\partial u_e}{\partial n}=(\sigma\nabla u)\cdot n on \Gamma_e \tag{5}$$

and satisfy the following matching conditions:

$$u_k = u_{k+1}, \sigma_k \frac{\partial u_k}{\partial n} = \sigma_{k+1} \frac{\partial u_{k+1}}{\partial n} \; at \, |x| = r_k, k = \overline{1, M-1} \tag{6}$$

on interfaces $|x| = r_k$. In the particular case when $\sigma_k = \sigma_b, k = 1,2,\dots,M$, so that the entire medium filling domain D is homogeneous and isotropic and, besides, $U_1 = const$, $U_2 = const$, the solution of (2)-(6) is described by formula (1).

Let us assume that the following conditions take place:

(i) $\sigma_k \in L^\infty(\Omega_k), \sigma_k \geq \sigma_k^0 = const > 0, k = 1,2,\dots,M$,

(ii) $U_1 \in H^{1/2}(\Gamma_1), U_2 \in H^{1/2}(\Gamma_2)$.

Here $H^{1/2}(\Gamma_1)$ and $H^{1/2}(\Gamma_2)$ are the well known trace spaces. It should be noted that conditions (i), (ii) ensure the existence and uniqueness of the weak solution to the direct problem (2)–(6). More concretely, the following theorem is valid.

Theorem 1. Let conditions (i), (ii) take place. Then problem (2)-(6) has a unique weak solution $(u_i, u_1, \dots, u_M, u_e) \in H^1(\Omega_i) \times H^1(\Omega_1) \times \dots \times H^1(\Omega_M) \times H^1(\Omega_e)$, which satisfies equations in (2), (3) in the distribution sense and boundary conditions in (4), (5) and (6) in the sense of traces. The following estimate holds for the solution $(u_i, u_1, \dots, u_M, u_e)$:

$$\|u_i\|_{H^1(\Omega_i)} + \sum_{k=1}^{M} \|u_k\|_{H^1(\Omega_k)} + \|u_e\|_{H^1(\Omega_e)} \leq C \left(¿ \vee U_1 \vee ¿_{H^{1/2}(\Gamma_1)} + \|U_2\|_{H^{1/2}(\Gamma_2)} \right).$$

Here C is a constant independent of U_1, U_2.

We remind that our goal is to solve inverse problems for the model (2)-(6) associated with designing shielding or cloaking shells and other functional devices for controlling static electric fields. Generally, the inverse problem consists of finding functions $\sigma_1, \sigma_2, \dots, \sigma_M$ from the two conditions

$$\nabla u_i = 0 \in \Omega_i, u_e = U^e \in \Omega_e. \tag{8}$$

Here, (u_i, u_e) is the restriction of the solution of problem (2)-(6) to $\Omega_i \times \Omega_e$. The shell $(\Omega, \sigma_1, \sigma_2, \dots, \sigma_M)$ which ensures the exact fulfillment of conditions (8) is called a perfect cloaking shell or simply a cloak. In the case when σ is determined solely from the first (or second) condition in (8) we will refer to the corresponding inverse problem as shielding (or external cloaking) problem.

3. Applying optimization method. Formulation of control problems

For solving our inverse problems we apply the optimization method. By this method the inverse problems are reduced to extremum problems of minimization of certain cost functionals which adequately correspond to the inverse problems of designing corresponding devices [22]. These functionals depend on conductivities $\sigma_1, \dots, \sigma_M$ of separate sublayers $(\underset{¿}{\overset{\Omega}{¿¿}} 1, \sigma_1), \dots, (\Omega_M, \sigma_M)$, of Ω. It is comfortable to define a variable vector $s = (\sigma_1, \sigma_2, \dots, \sigma_M)$, to which we will refer to as a conductivity vector of the shell (Ω, s).

In order to formulate our control problems, we denote by $U[s] = U[\sigma_1, \sigma_2, \dots, \sigma_M]$ the solution to problem (2)–(6) corresponding to (variable) conductivities σ_k in $\Omega_k, k = 1, \dots, M$, and to constant conductivity σ_b in Ω_i and Ω_e. We assume below that the vector $s=(\sigma_1,\sigma_2,\ldots,\sigma_M)$ belongs to the bounded set

$$S=\{s:0<\sigma_{min}\le\sigma_k\le\sigma_{max},k=1,2,\ldots,M\} \tag{9}$$

to which we refer to as a control set. Here given positive constants σ_{min} and σ_{max} are lower and upper bounds of the control set *S*. Let us define three cost functionals

$$J_i(s)=\frac{\|\nabla u_i[s]\|_{L^2(\Omega_i)}}{\|\nabla U^e\|_{L^2(\Omega_i)}},J_e(s)=\frac{\|u_e[s]-U^e\|_{L^2(\Omega_e)}}{\|U^e\|_{L^2(\Omega_e)}},J(s)=\alpha J_i(s)+\beta J_e(s)\ , \tag{10}$$

where α , $\beta\in[0,1]$ are nonnegative numbers satisfying $\alpha+\beta=1$,

$$\|U[s]-U^e\|^2_{L^2(\Omega_e)}=\int_{\Omega_e}|U[s]-U^e|^2dx,\|U^e\|^2_{L^2(\Omega_e)}=\int_{\Omega_e}|U^e|^2dx,$$

$$¿\vee\nabla u_i[s]\vee¿_{L^2(\Omega_i)}=\int_{\Omega_i}|\nabla u_i[s]|^2dx,$$

and formulate the following control problem:

$$J(s)=[\alpha J_i(s)+\beta J_e(s)]\to min\ ,\quad s\in S\ . \tag{11}$$

We note that in the particular case $\alpha=1$, $\beta=0$ (or $\alpha=0$, $\beta=1$) problem (11) corresponds to the shielding problem (or to the external cloaking problem). In another case when $\alpha=\beta=0.5$ it corresponds to the general cloaking problem.

Now we assume additionally that the following condition takes place:

(iii) $\sigma_k\in H^s(\Omega_k)$, $s>1$, $k=1,2,\ldots M$.

For theoretical study of general control problem (11) one can apply the mathematical procedure developed in [17] for solving inverse problems of mathematical physics by using optimization method. Based on this procedure we can prove the following theorem.

Theorem 2. Let conditions (ii), (iii) take place. Then problem (11), where the set S and functionals $J_i(s)$, $J_e(s)$ are defined in (9), (10), has at least one solution.

Now we consider the important particular case when all layers $\Omega_1,\Omega_2,\ldots,\Omega_M$ are filled with different homogeneous media. In this case all conductivities σ_k are constants and therefore problem (11) is finite dimensional. Moreover, one can show arguing as in [23] that in this case a cloaking performance of the cloak (Ω,s) is connected with a value $J(s)$ by inverse dependence: a smaller value $J(s)$ corresponds to a higher cloaking performance of the cloak (Ω,s) and vice versa. Therefore, our goal when solving problem (11) will consist of finding a conductivity vector (an optimal solution of (11)) $s^{opt}\in S$ for which functional J takes a minimum value $J^{opt}=J(s^{opt})$ on the set S and therefore the cloak (Ω,s^{opt}) possesses a maximum cloaking performance. For numerical solving problem (11) in this case we will use algorithm based on the particle swarm optimization (PSO) method [20].

4. Results of numerical simulations

In this section we discuss numerical results obtained when designing multilayer cloaking shells using PSO algorithm. Numerical simulation was performed for the following problem data: $a=1$ m, $b=2$ m, $\sigma_i=¿$ $\sigma_e=1.45\times10^6\sigma_0$ (corresponds to stainless steel) where $\sigma_0=1S/m$, $\sigma_{min}=1\times10^4S/m$ (corresponds to graphite), $\sigma_{max}=4.55\times10^7S/m$ (corresponds to gold). The role of the externally applied field U^e was played by the field (1).

The first group of tests is related to the shielding problem. Our optimization analysis using PSO algorithm for shielding problem showed that optimal values σ_K^{opt} of all parameters σ_K with odd indices $k=1,3,5,\ldots,M-1$ coincide with σ_{max} while optimal values $\sigma_2^{opt},\ldots,\sigma_M^{opt}$ (with even indices) coincide with σ_{min} . This means that the optimization design coincides with so-called alternating design (see details in [23]). The corresponding optimal values $J(s^{opt})$ where $s^{opt}=(\sigma_1^{opt},\sigma_2^{opt},\ldots,\sigma_M^{opt})$ of functional $J(s)$ are presented together with σ_M^{opt}/σ_0 and the values $J_e(s^{opt})$, $J(s^{opt})$ for comparison in Table I for different $M=2,4,\ldots,12$ specified in the first column.

It is seen from Table I that values $J_i(s^{opt})$ decrease from 8.85×10^{-3} to 1.16×10^{-7} when M varies from 2 to 12. The last value $J_i(s^{opt})$ = 1.16×10^{-7} for $M=12$ corresponds to very high shielding performance of the optimal shell (Ω, s^{opt}) . At the same time, the values $J_e(s^{opt})$ and $J(s^{opt})$ are large enough since we minimize just the functional $J_i(s)$. Presented results confirm the high performance of shielding shells obtained by using the optimization design for shells with a small number of homogeneous isotropic layers.

TABLE I

SIMULATION RESULTS FOR OPTIMIZED MULTILAYERED SHIELDS

$\sigma_{min}=1\times10^4\,S/m,$ ¿ $\sigma_{max}=4.55\times10^7\,S/m$¿

M σ_M^{opt}/σ_0 $J_i(s^{opt})$ $J_e(s^{opt})$ $J(s^{opt})$				
2	4.55×10^7	8.85×10^{-3}	1.43×10^{-1}	1.51×10^{-1}
4	4.55×10^7	2.2×10^{-3}	1.01×10^{-2}	1.03×10^{-1}
6	4.55×10^7	1.52×10^{-5}	7.35×10^{-2}	7.35×10^{-2}
8	4.55×10^7	2.01×10^{-6}	5.32×10^{-2}	5.32×10^{-2}
10	4.55×10^7	4.11×10^{-7}	3.82×10^{-2}	3.82×10^{-2}
12	4.55×10^7	1.16×10^{-7}	2.68×10^{-2}	2.68×10^{-2}

The second group of tests is related to the cloaking problem. Our optimization analysis showed that optimal values σ_k^{opt} obtained by using PSO algorithm for cloaking problem almost coincide with values σ_k^{alt} corresponding to alternating design for all j except $j=M$. The last optimal control σ_M^{opt} takes some intermediate value between σ_{min} and σ_{max} . The found values σ_M^{opt} are presented together with the corresponding values $J(s^{opt})$, $J_i(s^{opt})$ and

$J_e(s^{opt})$ where $s^{opt}=(\sigma_1^{opt},\sigma_2^{opt},\dots,\sigma_M^{opt})$ in Table II for different M equaled to 2, 4, 6, 8, 10 and 12.

In particular, it is seen from Table II that values $J(s^{opt})$ decrease from 3.1×10^{-2} to 3.08×10^{-7} when M varies from 2 to 12. The last value $J(s^{opt})$ = 3.08×10^{-7} corresponds to very high cloaking performance of the optimal shell (Ω, s^{opt}) . Thus, presented results confirm the high performance of cloaking shells obtained by using the optimization design for shells with a small number of layers.

TABLE II

SIMULATION RESULTS FOR OPTIMIZED MULTILAYERED CLOAKS

$\sigma_{min}=1\times10^4\,S/m,$ ¿ $\sigma_{max}=4.55\times10^7\,S/m$¿

M σ_M^{opt}/σ_0 $J_i(s^{opt})$ $J_e(s^{opt})$ $J(s^{opt})$				
2	5.09×10^6	3.1×10^{-2}	8.66×10^{-6}	3.1×10^{-2}
4	1.04×10^7	4.49×10^{-4}	1.21×10^{-6}	4.51×10^{-4}
6	1.55×10^7	2.41×10^{-5}	3.66×10^{-7}	2.45×10^{-5}
8	2.03×10^7	2.7×10^{-6}	3.85×10^{-6}	6.58×10^{-6}
10	2.49×10^7	5.08×10^{-7}	2.29×10^{-7}	7.38×10^{-7}
12	2.92×10^7	1.34×10^{-7}	1.73×10^{-7}	3.08×10^{-7}

5. Conclusion

We studied control problems for electric conductivity model (2)–(6). These problems arise when optimization method is applied for solving static electric field shielding and cloaking problems. The electric conductivities of the shell layers play the role of passive controls. We have proved the theorem about correct solvability of the direct conductivity problem and proved solvability of general control problem. We also proposed numerical algorithm for solving our control problems which is based on particle swarm optimization method. Optimization analysis showed that a high shielding or cloaking performance of the designed devices can be achieved when using multilayer shell consisting of several isotropic homogeneous layers with optimal constant conductivities. We emphasize that high cloaking and shielding performances can be achieved without use of methamaterials, but using natural materials with high contrast.

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