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Gennady Alekseev[1,2], Yuliya Spivak[1,2], Aleksey Lobanov[1]
1) Institute of Applied Mathematics FEB RAS, Vladivostok, Russia
2) Far Eastern Federal University, Vladivostok, Russia
alekseev@iam.dvo.ru
uliyaspivak@gmail.com
Alekslobanov1@mail.ru

CONTROL APPROACH IN 3D PROBLEMS OF DESIGNING OF BILAYERED MAGNETIC CLOAKS

Abstract. We consider control problems for the 3D model of magnetic scattering by a permeable isotropic obstacle having the form of a spherical bilayer shell. These problems arise while developing the design technologies of magnetic cloaking devices using the optimization method for solving corresponding inverse problems. The solvability of direct and optimization problems for the magnetic scattering model under study is proved. The optimality system which describes the necessary conditions of extremum is derived. Also, numerical aspects of applying the optimization approach for solving problems of designing magnetic cloaking devices are discussed.

Key words magnetic cloaking, inverse problem, optimization method, existence, optimality system.

1. Introduction

In the last few years, devices for cloaking material objects have attracted the rapid attention in the research fields of “invisibility” and metamaterials. The first works in the mentioned fields were the articles [1, 2, 3]. These were the start of development of different methods, schemes and techniques of solving problems when cloaking electromagnetic waves, acoustic waves, magnetic, electric, thermal and other static fields [4, 5, 6, 7]. It should be noted that the solutions obtained in these papers possess several drawbacks. In particular, some components of spatially dependent parameter tensors of ideal cloak are required to have infinite or zero values at the inner boundary of the cloak [2] which are very difficult to implement.

One of approaches of overcoming these difficulties consists of replacing “exact” singular solutions of cloaking problems under study by approximate non-singular solutions and designing cloaking devices based on these approximations (see., e.g., [8, 9, 10, 11]). Alternative approach is based on using the optimization method of solving inverse problems. This method is based on replacing initial cloaking problem by minimization problem of a suitable tracking-type cost functional which corresponds to inverse problem under consideration. The method to which one should referred to as inverse design method [12, 13] was applied in [14] devoted to the numerical analysis of 2D problems of designing layered cloaking shells. Also, it was applied in [15]–[21] when studying theoretically electromagnetic or acoustic cloaking problems. Paper [22] are devoted to studying invisibility problem in X-ray tomography.

The goal of this paper is theoretical analysis of the cloaking problem for the 3D model of magnetic scattering by bilayer shell using the optimization method. The plan of our paper is as follows. Firstly, we formulate the direct magnetic scattering problem and prove its correct solvability. Then we formulate the corresponding inverse cloaking problem, and reduce it to respective control problem for which we prove the solvability. Finally, we discuss numerical aspects of applying the optimization approach for solving problems of designing magnetic cloaking devices under study.

2. Statement and analysis of direct magnetic scattering problem

We consider the domain in the space R^3 having in spherical coordinates (r,θ,φ) the form of spherical layer $\Omega_1=\{\mathbf{x}\in R^3: a<r=|\mathbf{x}|<b\}$ surrounded by another spherical layer

$\Omega_2 = \{\mathbf{x} \in R^3 : b < r = |\mathbf{x}| < c\}$, where a, b and c are positive constants. Denote by Ω_i and Ω_e^∞ the interior and exterior of the domain $\Omega_1 \cup \Omega_2$. We will assume that the domains Ω_i and Ω_e^∞ are filled by homogeneous isotropic media with constant permeabilities μ_i and μ_e, respectively, while Ω_1 and Ω_2 are filled by inhomogeneous isotropic media with variable permeabilities μ_1 and μ_2, respectively (see Fig. 1).

Denote by B_R the ball $|\mathbf{x}| < R$, where $R > c$, containing domains Ω_i, Ω_1 and Ω_2. We will assume that there are sources outside B_R, which generate externally applied magnetic field $\mathbf{H}_a = -\operatorname{grad}\Phi_a$ corresponding to potential Φ_a which satisfies the Laplace equation $\Delta\Phi_a = 0$ in B_R.

$$\mu_i \Delta\Phi_i = 0 \text{ in } \Omega_i, \quad \operatorname{div}(\mu_1 \nabla\Phi_1) = 0 \text{ in } \Omega_1, \quad \operatorname{div}(\mu_2 \nabla\Phi_2) = 0 \text{ in } \Omega_2, \quad \mu_e \Delta\Phi_e = 0 \text{ in } \Omega_e^\infty, \tag{1}$$

$$\Phi_i = \Phi_1, \quad \mu_i \frac{\partial\Phi_i}{\partial r} = \mu_1 \frac{\partial\Phi_1}{\partial r} \text{ at } r = a, \quad \Phi_1 = \Phi_2, \quad \mu_1 \frac{\partial\Phi_1}{\partial r} = \mu_2 \frac{\partial\Phi_2}{\partial r} \text{ at } r = b, \tag{2}$$

$$\Phi_2 = \Phi_a + \Phi_s, \quad \mu_2 \frac{\partial\Phi_2}{\partial r} = \mu_e \frac{\partial(\Phi_a + \Phi_s)}{\partial r} \text{ at } r = c, \quad \Phi_s(\mathbf{x}) = o(1) \quad \text{as} \quad r = |\mathbf{x}| \to \infty. \tag{3}$$

In the particular case, when μ_1 and μ_2 are nonnegative constants and, besides, the field $\mathbf{H}_a$ is uniform, the direct problem (1)–(3) admits an exact solution which can be found using Fourier's method (see [9, 10]).

Now, consider the case when the external field $\mathbf{H}_a$ is inhomogeneous and therefore it is impossible to apply Fourier's method for finding an exact solution of the problem (1)–(3). A number of functional spaces will be used while studying the direct problem (1)–(3) and respective control problems. Let $\Omega_e = \Omega_e^\infty \cap B_R$. We will use the space $H^1(D)$, where D is one of domains B_R, Ω_i, Ω_1, Ω_2 Ω_e^∞, and also spaces $L^\infty(\Omega_k)$, $H^s(\Omega_k)$, $s > 0$, $k = 1,2$, $L^2(Q)$, $H^{1/2}(\Gamma_R)$ and $H^{-1/2}(\Gamma_R)$. Here $Q \subset B_R$ is an arbitrary open subset of B_R, Γ_R is a boundary of B_R (see Fig. 2).

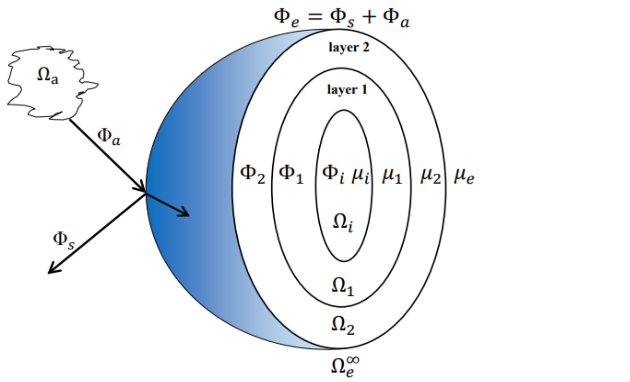

Figure 1: Layout of external sources and spherical bilayer cloak

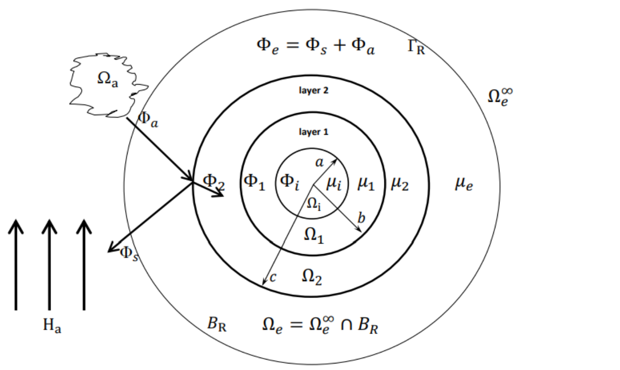

Figure 2: Schematic layout of artificial boundary Γ_R

The norms and scalar products in $H^1(D)$, $H^s(\Omega_k)$, $L^2(Q)$, $H^{1/2}(\Gamma_R)$ and $H^{-1/2}(\Gamma_R)$ will be denoted by $\|\cdot\|_{1,D}$, $(\cdot,\cdot)_{1,D}$, $\|\cdot\|_{s,\Omega_k}$, $(\cdot,\cdot)_{s,\Omega_k}$, $\|\cdot\|_Q$, $(\cdot,\cdot)_Q$, $\|\cdot\|_{1/2,\Gamma_R}$ and $\|\cdot\|_{-1/2,\Gamma_R}$. We set $L^\infty_{\lambda_0}(\Omega_k) = \{\lambda \in L^\infty(\Omega_k) : \lambda(\mathbf{x}) \geq \lambda_0\}$, $H^s_{\lambda_0}(\Omega_k) = \{\lambda \in H^s(\Omega_k) : \lambda(\mathbf{x}) \geq \lambda_0\}$, $\lambda_0 = \text{const} > 0$, $k = 1,2$. It is well known by the embedding theorem that the continuous and compact embedding $H^s(\Omega_k) \subset L^\infty(\Omega_k)$ at $s > 3/2$, $k = 1,2$, holds and the following estimate takes place:

$$\|\lambda\|_{L^\infty(\Omega_k)} \leq C_s \|\lambda\|_{s,\Omega_k} \quad \forall \lambda \in H^s(\Omega_k), \, s > 3/2, \, k = 1,2. \tag{4}$$

Here C_s is a constant depending on $s > 3/2$ and Ω_1, Ω_2. We need also a subspace $H(\Omega_e) = \{\Phi \in H^1(\Omega_e) : \Delta\Phi = 0 \text{ in } \Omega_e\}$, equipped with the norm $\|\cdot\|_{1,\Omega_e} = \|\cdot\|_{H^1(\Omega_e)}$. The space $H(\Omega_e)$ will be served for describing restrictions of externally applied field Φ_a to the domain Ω_e.

It should be noted that by the trace theorem there exists a trace $\Phi|_{\Gamma_R} \in H^{1/2}(\Gamma_R)$ for any function $\Phi \in H^1(B_R)$ while for any function $\Phi^e \in H(\Omega_e)$ there exists a normal trace $\partial\Phi^e/\partial n|_{\Gamma_R}$ and the following estimates hold:

$$\|\Phi\|_{1/2,\Gamma_R} \le C_R \|\Phi\|_X \;\; \forall \Phi \in H^1(B_R), \tag{5}$$

$$\|\partial\Phi^e/\partial n\|_{-1/2,\Gamma_R} \le C_R' \|\Phi^e\|_{1,\Omega_e} \;\; \forall \Phi^e \in H(\Omega_e). \tag{6}$$

Here C_R, C_R' are constants depending on Ω_e and R but are independent of $\Phi \in X$ and $\Phi^e \in H(\Omega_e)$.

We assume below that the following conditions take place:

(i) $\mu_1 \in L^\infty_{\mu_1^0}(\Omega_1)$, $\mu_2 \in L^\infty_{\mu_2^0}(\Omega_2)$, $\mu_1 \ge \mu_1^0$, $\mu_2 \ge \mu_2^0$, $\mu_1^0 = \text{const} > 0$, $\mu_2^0 = \text{const} > 0$;

(ii) $\Phi^e \equiv \Phi_a|_{\Omega_e} \in H(\Omega_e)$.

As the potential Φ is determined up to an additive constant we will not distinguish functions of the space $H^1(B_R)$ which differ from each other by an additive constant. Thus the main role below will be played by the following quotient-space $X = H^1(B_R)$ with the norm:

$$\|\Phi\|_X^2 = \|\nabla\Phi\|_{\Omega_i}^2 + \|\nabla\Phi\|_{\Omega_1}^2 + \|\nabla\Phi\|_{\Omega_2}^2 + \|\nabla\Phi\|_{\Omega_e}^2. \tag{7}$$

One can show that the space $X = H^1(B_R)$ is Hilbert for this norm and, besides, the following analogue of Poincare-Friedrichs inequality holds:

$$\|\Phi\|_{B_R} \le C_P \|\Phi\|_X \;\; \forall \Phi \in X. \tag{8}$$

Here C_P is a constant which is independent of $\Phi \in X$.

We begin our analysis with defining weak formulation and weak solution of direct problem (1)–(3). Preliminarily we reduce problem (1)–(3) to an equivalent boundary problem considered in the bounded domain (ball) B_R. To this end we introduce as in [19] the Dirichlet-to-Neumann operator $T : H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R)$ which maps every function $h \in H^{1/2}(\Gamma_R)$ to the function $\partial\tilde{\Phi}/\partial\nu \in H^{-1/2}(\Gamma_R)$. Here $\tilde{\Phi}$ is a solution of the external Dirichlet problem for equation $\Delta\tilde{\Phi} = 0$ in $\Omega_e^\infty \setminus B_R$ with the boundary condition $\tilde{\Phi}|_{\Gamma_R} = h$ satisfying the condition $\tilde{\Phi}(\mathbf{x}) = o(1)$ as $r = |\mathbf{x}| \to \infty$. We note that problem (1)–(3) considered in R^3 is equivalent to boundary value problem (1), (2) considered in the ball B_R under the following additional condition for Φ_s on Γ_R:

$$\partial\Phi_s/\partial n = T\Phi_s \text{ on } \Gamma_R \tag{9}$$

For brevity we will refer below to the problem (1), (2), (9) as Problem 1.

Based on the space X we derive now the weak formulation of Problem 1. Let $S \in X$ be a test function. We multiply every of equations in (1) considered in domains Ω_i, Ω_1, Ω_2 and Ω_e, by S, integrate over Ω_i, Ω_1, Ω_2 or Ω_e, respectively, and apply Green formulae. Adding the obtained identities and using the boundary conditions in (2) and (9) we arrive at the following identity for the quadruple $\Phi = (\Phi_i, \Phi_1, \Phi_2, \Phi_e) \in X$:

$$a_\mu(\Phi, S) \equiv a_0(\Phi, S) + a(\mu_1, \mu_2; \Phi, S) = \langle F, S \rangle \;\; \forall S \in X. \tag{10}$$

Here and below μ denotes the pair (μ_1, μ_2) while $a_0(\cdot,\cdot)$, $a(\mu_1, \mu_2; \cdot,\cdot)$ and F are bilinear and linear forms defined by

$$a_0(\Phi,S)=\mu_i\int_{\Omega_i}\nabla\Phi\cdot\nabla S d\mathbf{x}+\mu_e\int_{\Omega_e}\nabla\Phi\cdot\nabla S d\mathbf{x}-\int_{\Gamma_R}(T\Phi)S d\sigma, \tag{11}$$

$$a(\mu_1,\mu_2;\Phi,S)=a_1(\mu_1;\Phi,S)+a_2(\mu_2;\Phi,S),\ \langle F,S\rangle=-\int_{\Gamma_R}T\Phi^e S d\sigma+\int_{\Gamma_R}(\partial\Phi^e/\partial n)S d\sigma \tag{12}$$

$$a_1(\mu_1;\Phi,S)=\int_{\Omega_1}\mu_1\nabla\Phi\cdot\nabla S d\mathbf{x},\ a_2(\mu_2;\Phi,S)=\int_{\Omega_2}\mu_2\nabla\Phi\cdot\nabla S d\mathbf{x}, \tag{13}$$

Identity (10) represents the weak formulation of problem (1), (2), (9) and its solution $\Phi=(\Phi_i,\Phi_1,\Phi_2,\Phi_e)\in X$ will be called a weak solution of Problem 1. Arguing as in [19], one can easily show, that the introducing of the weak solution is admissible in the following sense: it satisfies all equations in (1) in the distribution sense, and also boundary conditions in (2) and (9) in the trace sense.

Using Hölder inequality and definition (7) for the norm $\|\cdot\|_X$ and (9), (10) we have

$$\left|\int_{\Omega_1}\mu_1\nabla\Phi\cdot\nabla S d\mathbf{x}\right|\le\|\mu_1\|_{L^\infty(\Omega_1)}\|\nabla\Phi\|_{\Omega_1}\|\nabla S\|_{\Omega_1}\le\|\mu_1\|_{L^\infty(\Omega_1)}\|\Phi\|_X\|S\|_X,$$

$$\left|\int_{\Omega_2}\mu_2\nabla\Phi\cdot\nabla S d\mathbf{x}\right|\le\|\mu_2\|_{L^\infty(\Omega_2)}\|\nabla\Phi\|_{\Omega_2}\|\nabla S\|_{\Omega_2}\le\|\mu_2\|_{L^\infty(\Omega_2)}\|\Phi\|_X\|S\|_X.$$

Besides, using (5), (6), we derive from (11), (12) and (13) that

$$\left|\mu_i\int_{\Omega_i}\nabla\Phi\cdot\nabla S d\mathbf{x}\right|\le\mu_i\|\Phi\|_X\|S\|_X,\ \left|\mu_e\int_{\Omega e}\nabla\Phi\cdot\nabla S d\mathbf{x}\right|\le\mu_e\|\Phi\|_X\|S\|_X, \tag{14}$$

$$\left|\int_{\Gamma_R}(T\Phi)S d\sigma\right|\le\|T\Phi\|_{-1/2,\Gamma_R}\|S\|_{1/2,\Gamma_R}\le C_T C_R^2\|\Phi\|_X\|S\|_X, \tag{15}$$

$$|a_0(\Phi,S)|\le(\mu_0+C_T C_R^2)\|\Phi\|_X\|S\|_X,\ \mu_0=\max(\mu_i,\mu_e), \tag{16}$$

$$|\langle F,S\rangle|\le\left(\|T\|\|\Phi^e\|_{1/2,\Gamma_R}+\|\partial\Phi^e/\partial n\|_{-1/2,\Gamma_R}\right)\|S\|_{1/2,\Gamma_R}\le(C_T C_R+C_R')C\|\Phi^e\|_{1,\Omega_e}\|S\|_X. \tag{17}$$

It follows from these estimates and (12) that forms $a(\mu_1,\mu_2;\cdot,\cdot)$, a_0 and F are continuous on X, and the following estimates hold:

$$|a(\mu_1,\mu_2;\Phi,S)|\le\left(\|\mu_1\|_{L^\infty(\Omega_1)}+\|\mu_2\|_{L^\infty(\Omega_2)}\right)\|\Phi\|_X\|S\|_X, \tag{18}$$

$$\|a_0\|\le C_0,\ \|F\|_{X^*}\le C_0\|\Phi^e\|_{1,\Omega_e},\ C_0=\max\left[\mu_0+C_T C_R^2,(C_T C_R+C_R')C_R\right]. \tag{19}$$

Here X^* is a dual of X with respect to space $L_0^2(B_R)$.

If, moreover, the condition (i) takes place then we have at $\mu_0=\min(\mu_i,\mu_e)$:

$$a_0(\Phi,\Phi)=\mu_i\int_{\Omega_i}|\nabla\Phi|^2 d\mathbf{x}+\mu_e\int_{\Omega_e}|\nabla\Phi|^2 d\mathbf{x}-\int_{\Gamma_R}(T\Phi)\Phi d\sigma\ge\mu_0\int_{\Omega_0\cup\Omega_e}|\nabla\Phi|^2 d\mathbf{x}\ \forall\Phi\in X,$$

$$\int_{\Omega_1}\mu_1\nabla\Phi\cdot\nabla\Phi d\mathbf{x}\ge\mu_1^0\|\nabla\Phi\|_{\Omega_1}^2,\ \int_{\Omega_2}\mu_2\nabla\Phi\cdot\nabla\Phi d\mathbf{x}\ge\mu_2^0\|\nabla\Phi\|_{\Omega_2}^2.$$

It follows from these estimates and the definition of the norm (7) in space X that the bilinear form a_μ defined in (10) is coercive in X under condition (i) and, besides,

$$a_\mu(\Phi,\Phi)\ge\mu_*\|\Phi\|_X^2\ \ \forall\Phi\in X,\ \mu_*=\min(\mu_i,\mu_e,\mu_1^0,\mu_2^0). \tag{20}$$

We note that the bilinear form a_μ introduced in (10) defines a linear operator $A_\mu:X\to X^*$ acting by

$$\langle A_\mu\Phi,S\rangle=a_\mu(\Phi,S)\equiv a_0(\Phi,S)+a(\mu_1,\mu_2;\Phi,S), \tag{21}$$

while problem (10) is equivalent to operator equation

$$A_\mu\Phi=F. \tag{22}$$

As the bilinear form a_μ is continuous and coercive on X under condition (i), then from the Lax-Milgram theorem follows that the operator $A_\mu:X\to X^*$ is an isomorphism and the inverse

$A_{\mu}^{-1}: X \to X^*$ to operator A_μ is also isomorphism. Let $C_\mu = \left\|A_\mu^{-1}\right\|$. It is clear by the Lax-Milgram theorem that $C_\mu \le C_1 = (1/\mu_*)$. Using this estimate we derive that operator equation (22) has for any element $F \in X^*$ a unique solution $\Phi_\mu = A_\mu^{-1}(F) \in X$, for which the estimate $\left\|\Phi_\mu\right\|_X \le C_1 \|F\|_{X^*}$ holds. Using this fact and the second estimate in (19) we conclude that for any field $\Phi^e \in H(\Omega_e)$ problem (10) has a unique solution $\Phi_\mu \in X$, and the following estimate holds:

$$\left\|\Phi_\mu\right\|_X \le C_2 \left\|\Phi^e\right\|_{1,\Omega_e}, C_2 = C_0 C_1. \tag{23}$$

We emphasize that constants C_1 and C_2 in (23) depend on μ_i, μ_e, μ_1^0, μ_2^0 and R but do not depend on functions μ_1 and μ_2 satisfying condition (i). Let us formulate the results obtained as the following theorem.

Theorem 1. *Let conditions (i), (ii) take place. Then for any pair* $\mu = (\mu_1, \mu_2)$ *we have:*

1) the operator $A_\mu : X \to X^*$ *defined in (21) is an isomorphism and for the inverse operator* $A_{\mu}^{-1}: X \to X^*$ *the estimate* $\left\|A_\mu^{-1}\right\| \le C_1 \equiv (1/\mu_*)$ *holds where constant* μ_* *is defined in (20);*

2) for any external field $\Phi^e \in H(\Omega_e)$ *problem (10) has a unique solution* $\Phi_\mu \in X$*, and the estimate (23) holds where the constant* C_2 *depends on* μ_i, μ_e, μ_1^0, μ_2^0 *and* R *but is independent of* (μ_1, μ_2).

3. Formulation of the inverse problem. Applying the optimization method. Derivation of the optimality system

In this section we consider the inverse problem for model (1)–(3) arising while developing the design technologies of magnetic cloaking devices. This problem consists of finding unknown permeabilities μ_1 and μ_2 of inhomogeneous media filling Ω_1 and Ω_2 from the following conditions $\nabla\Phi = 0$ in Ω_i, $\Phi = \Phi_a$ in Ω_e. For solving this inverse problem we apply the optimization method which is based on minimization of certain cost functional. As a cost functional we choose the following:

$$I(\Phi) = [\alpha I_1(\Phi) + \beta I_2(\Phi)] \quad (\alpha + \beta = 1), \tag{24}$$

where

$$I_1(\Phi) = \|\nabla\Phi\|_{\Omega_i}^2 \equiv \int_{\Omega_i} |\nabla\Phi|^2 d\mathbf{x}, \quad I_2(\Phi) = \|\Phi - \Phi_a\|_{\Omega_e}^2 \equiv \int_{\Omega_e} |\Phi - \Phi_a|^2 d\mathbf{x}. \tag{25}$$

Here α, $\beta \in [0,1]$ are arbitrary parameters satisfying condition $\alpha + \beta = 1$. The choice of functional $I(\Phi)$ responds to the problem of internal (or external) cloaking at $\alpha = 1$, $\beta = 0$ (or at $\alpha = 0$, $\beta = 1$). For the remaining values of α and β functional (25) corresponds to the general cloaking problem.

We will assume that the controls μ_1 and μ_2 are changed over certain sets K_1 and K_2. It is assumed that the following conditions hold.

(j) $K_1 \subset H_{\mu_1^0}^s(\Omega_1)$, $\mu_1^0 = \text{const} > 0$, $K_2 \subset H_{\mu_2^0}^s(\Omega_2)$, $\mu_2^0 = \text{const} > 0$, $s > 3/2$; $\alpha_0 > 0$.

Setting $K = K_1 \times K_2$, $\mu = (\mu_1, \mu_2)$ we define the operator $G : X \times K \to X^*$ by

$$\langle G(\Phi, \mu), S\rangle = \langle A_\mu \Phi, S\rangle - \langle F, S\rangle \equiv a_0(\Phi, S) + a(\mu_1, \mu_2; \Phi, S) - \langle F, S\rangle \; \forall S \in X \tag{26}$$

and rewrite the weak formulation (10) of problem 1 as $G(\Phi, \mu) = 0$. Consider the following control problem:

$$J(\Phi, \mu) \equiv \frac{\alpha_0}{2} I(\Phi) + \frac{\alpha_1}{2}\|\mu_1\|_{s,\Omega_1}^2 + \frac{\alpha_2}{2}\|\mu_2\|_{s,\Omega_2}^2 \to \inf, G(\Phi, \mu) = 0, (\Phi, \mu) \in X \times K. \tag{27}$$

Here α_0, α_1, α_2 are non-negative parameters which are used to control the relative importance of terms in (28). Set $Z_{ad} = \{(\Phi,\mu)\in X\times K : G(\Phi,\mu)=0, J(\Phi,\mu)<\infty\}$.

Theorem 2. *Let, under assumptions (ii) and (j),* $\alpha_1>0$, $\alpha_2>0$ *or* $\alpha_1\ge 0$, $\alpha_2\ge 0$ *and* K_1, K_2 *be bounded sets. Then control problem (26) has at least one solution* $(\Phi,\mu_1,\mu_2)\in X\times K_1\times K_2$.

Proof. Denote by $(\Phi^m,\mu^m)\in Z_{ad}$, $\mu^m=(\mu_1^m,\mu_2^m)$, $m\in N\equiv\{1,2,...\}$ a minimizing sequence for the functional J for which the following relations hold:

$$a_0(\Phi^m,S)+a(\mu_1^m,\mu_2^m;\Phi^m,S)=\langle F,S\rangle\ \forall S\in X,\ \lim_{m\to\infty}J(\Phi^m,\mu_m)=\inf_{(\Phi,\mu)\in Z_{ad}}J(\Phi,\mu)\equiv J^*. \quad (28)$$

By conditions of Theorem 2 we have

$$\|\mu_1^m\|_{s,\Omega_1}\le c_1,\quad \|\mu_2^m\|_{s,\Omega_2}\le c_2\ \forall m\in N. \quad (29)$$

Here and below c_1, c_2, c_3 are some constants which do not depend on m. It follows from (29) and Theorem 1 that $\|\Phi^m\|_X\le c_3$. From this estimate and (29) we derive that there are weak limits $\mu_1^*\in K_1$, $\mu_2^*\in K_2$, $\Phi^*\in X$ of some subsequences of sequences $\{\mu_1^m\}$, $\{\mu_2^m\}$, $\{\Phi^m\}$. Using this fact and compactness of embedding $H^s(\Omega_k)\subset L^\infty(\Omega_k)$ at $s>3/2$, $k=1,2$ we conclude that $\mu_1^m\to\mu_1^*$ strongly in $L^\infty(\Omega_1)$, $\mu_2^m\to\mu_2^*$ strongly in $L^\infty(\Omega_2)$, while $\Phi^m\to\Phi^*$ weakly in X.

Let us prove that $G(\Phi^*,\mu_1^*,\mu_2^*)=0$, i.e. that

$$a_0(\Phi^*,S)+a(\mu_1^*,\mu_2^*;\Phi^*,S)=\langle F,S\rangle\ \forall S\in X. \quad (30)$$

To this end we pass to the limit in (28) as $m\to\infty$. It is clear that linear term $a_0(\Phi^m,S)$ passes to $a_0(\Phi^*,S)$ as $m\to\infty$ while for the difference $a(\mu_1^m,\mu_2^m;\Phi^m,S)-a(\mu_1^*,\mu_2^*;\Phi^*,S)$ we have by (12)

$$\left|a(\mu_1^m,\mu_2^m;\Phi^m,S)-a(\mu_1^*,\mu_2^*;\Phi^*,S)\right|\le\left|a_1(\mu_1^m;\Phi^m,S)-a_1(\mu_1^*;\Phi^*,S)\right|+\left|a_2(\mu_2^m;\Phi^m,S)-a_2(\mu_2^*;\Phi^*,S)\right|. \quad (31)$$

Let us show that every of terms in the right-hand side of (31) vanishes as $m\to\infty$. In fact, taking into consideration (13), we have for the first term in (31):

$$\left|a_1(\mu_1^m;\Phi^m,S)-a_1(\mu_1^*;\Phi^*,S)\right|\le\left|\int_{\Omega_1}\mu_1^*(\nabla\Phi^m-\nabla\Phi^*)\cdot\nabla S d\mathbf{x}\right|+\left|\int_{\Omega_1}(\mu_1^m-\mu_1^*)\nabla\Phi^m\cdot\nabla S d\mathbf{x}\right|\ \forall S\in X. \quad (32)$$

As $\Phi^m\to\Phi^*$ weakly in X then the first integral in the right-hand side of (32) tends to zero as $m\to\infty$ for any $S\in X$. Besides, it follows from the strong convergence $\mu_1^m\to\mu_1^*$ in $L^\infty(\Omega_1)$ that the second integral in (32) also tends to zero as $m\to\infty$. This means that the first term in the right-hand side of (31) tends to zero as $m\to\infty$. In a similar way one can show using (14) that the second term in the right-hand side of (31) tends to zero as $m\to\infty$. Therefore passing to the limit in (28) as $m\to\infty$ we arrive at (29). This means that $G(\Phi^*,\mu^*)=0$ where $\mu^*=(\mu_1^*,\mu_2^*)$. As the functional $J(\Phi)$ is a weakly lower semicontinuous functional on $X\times K$ we derive that $J(\Phi^*,\mu^*)=J^*$. This proves the theorem.

The second stage of analysis of extremum problem (27) consists of derivation an optimality system describing necessary conditions of extremum. For this purpose we make use of the extremum principle in smoothly-convex extremum problems [23]. Preliminarily we find the Fréchet derivative with respect to Φ of the operator $G: X\times K\to X^*$ defined in (26). It follows from linearity of the operator G with respect to Φ that the Fréchet derivative $G'_\Phi(\hat\Phi,\hat\mu)$ at every point $(\hat\Phi,\hat\mu)\in X\times K$ where $\hat\mu=(\hat\mu_1,\hat\mu_2)$ is defined by $G'_\Phi(\hat\Phi,\hat\mu)=\hat A\equiv A_{\hat\mu}$. Here operator $A_{\hat\mu}$ is defined by (21) at $\mu=\hat\mu$. Besides, it follows from (24) that

$$\langle (I')_\Phi(\hat\Phi),S\rangle=\alpha(\nabla\hat\Phi,\nabla S)_{\Omega_i}+\beta(\hat\Phi-\Phi_a,S)_{\Omega_e}\ \forall S\in X. \quad (33)$$

Following [32] we introduce a Lagrange multiplier $\Psi \in X$ which will be referred to as an adjoint stage and consider the Lagrangian $L: X \times K \times X \to R$ defined by the formula $L(\Phi,\mu,\Psi) \equiv J(\Phi,\mu) + \langle G(\Phi,\mu),\Psi\rangle_{X^* \times X}$. Denote by $\hat{A}^* \equiv G'_\Phi(\hat{\Phi},\hat{\mu})^* : X \to X^*$ the operator adjoint for operator $\hat{A}^* \equiv G'_\Phi(\hat{\Phi},\hat{\mu}): X \to X^*$ defined by

$$\left\langle \hat{A}^*\Psi, S\right\rangle_{X^* \times X} \equiv \left\langle G'_\Phi(\hat{\Phi},\hat{\mu})^*\Psi, S\right\rangle_{X^* \times X} = \left\langle G'_\Phi(\hat{\Phi},\hat{\mu})S,\Psi\right\rangle_{X^* \times X} = \left\langle \hat{A}S,\Psi\right\rangle_{X^* \times X} \ \forall \Psi, S \in X .$$

It follows from linearity of operator G with respect to μ_1, μ_2 and from convexity of the set $K = K_1 \times K_2$ that the set $G(\Phi,K) = \{\mathbf{x}^* = G(\Phi,\mu) \in X^*, \mu \in K\}$ is a convex subset of X^* for any function $\Phi \in X$. As the operator $G'_\Phi(\hat{\Phi},\hat{\mu}) \equiv \hat{A}$ is an isomorphism by Theorem 1, then from results of [23] the next theorem follows.

Theorem 3. *Let under assumptions (ii) and (j) the pair* $(\hat{\Phi},\hat{\mu}) \in X \times K$ *where* $\hat{\mu} = (\hat{\mu}_1,\hat{\mu}_2)$ *be a solution of problem (27). Then there exists a unique Lagrange multiplier* $\hat{\Psi} \in X$ *that satisfies the Euler-Lagrange equation*

$$a_{\hat{\mu}}(S,\hat{\Psi}) \equiv a_0(S,\hat{\Psi}) + a(\hat{\mu}_1,\hat{\mu}_2;S,\hat{\Psi}) = -\alpha_0\left[\alpha(\nabla\hat{\Phi},\nabla S)_{\Omega_i} + \beta(\hat{\Phi}-\Phi_a,S)_{\Omega_e}\right] \quad \forall S \in X \quad (34)$$

and the minimum principle holds, which is equivalent to the following variational inequalities:

$$\alpha_1(\hat{\mu}_1,\mu_1-\hat{\mu}_1)_{s,\Omega_1} + a_1(\mu_1-\hat{\mu}_1,\hat{\Phi},\hat{\Psi}) \geq 0 \ \forall \mu_1 \in K_1, \quad (35)$$

$$\alpha_2(\hat{\mu}_2,\mu_2-\hat{\mu}_2)_{s,\Omega_2} + a_2(\mu_2-\hat{\mu}_2,\hat{\Phi},\hat{\Psi}) \geq 0 \ \forall \mu_2 \in K_2 . \quad (36)$$

We note that the direct problem (10), where one should put $\Phi = \hat{\Phi}$, $\mu = \hat{\mu}$, adjoint problem (34) with respect to the adjoint field $\hat{\Psi}$ and the inequalities (35), (36) represent the optimality system describing the necessary extremum conditions for the problem (27). The optimality system plays an important role when solving control problems. Firstly, it can be used to establish sufficient conditions for the data that ensure the uniqueness and stability of the optimal solutions to the considered control problems with respect to small perturbations of the data (see, for example, [24]).

Secondly, the constructed optimality system can be used to develop an efficient numerical algorithm for solving control problem (27). The simplest algorithm can be obtained using the simple iteration method. The m-th iteration of such an algorithm consists in finding unknown values of Φ^m, Ψ^m, μ_r^{m+1} and μ_θ^{m+1} given μ_r^m, μ_θ^m, $m = 1,2,...,$, starting with the initial data μ_r^0, μ_θ^0 by successively solving the following problems:

$$\begin{gathered} a_0(\Phi^m,S) + a_1(\mu_r^m;\Phi^m,S) + a_2(\mu_\theta^m;\Phi^m,S) = \langle F,S\rangle \quad \forall S \in X , \\ a_0(S,\Psi^m) + a_1(\mu_r^m;S,\Psi^m) + a_2(\mu_\theta^m;S,\Psi^m) = \\ = \alpha_0\left[\alpha\int_{\Omega_i}\nabla\Phi^m\cdot\nabla S d\mathbf{x} + \beta\int_{\Omega_e}(\Phi^m-\Phi_a)S d\mathbf{x}\right] \quad \forall S \in X , \\ \alpha_1(\mu_r^{m+1},\mu_r-\mu_r^m)_{s,\Omega} + a_1((\mu_r-\mu_r^{m+1})\Phi^m,\Psi^m) \geq 0 \quad \forall \mu_r \in K_1, \\ \alpha_2(\mu_\theta^{m+1},\mu_\theta-\mu_\theta^m)_{s,\Omega} + a_2((\mu_\theta-\mu_\theta^{m+1})\Phi^m,\Psi^m) \geq 0 \quad \forall \mu_\theta \in K_2 . \end{gathered} \quad (37)$$

An alternative algorithm is based on the use of the particle swarm method according to the scheme proposed in [27, 28] for the numerical solution of 2D problem of thermal cloaking.

4. Conclusion

In conclusion, we studied control problems for magnetic scattering model (1)–(3). These problems arise when optimization method is applied for solving cloaking problems for respective scattering model. The magnetic permeabilities μ_1 and μ_2 of the inhomogeneous media filling the layers Ω_1 and Ω_2 of the cloak play the role of controls. We have proved the correctness of direct magnetic scattering problem and proved solvability of general control problem (28). Also, we derived the optimality system describing the necessary conditions of extremum and proposed two numerical algorithms for solving our cloaking problems. Authors plan to devote a forthcoming paper to developing and studying the properties of the mentioned algorithms and to comparative analysis of results of numerical experiments performed using these algorithms.

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