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ON RECOVERING OF DIFFUSION COEFFICIENT FOR GENERILIZED BOUSSINESQ MODEL

Abstract. Boundary value and inverse coefficient problems for a generalized Oberbeck-Boussinesq model are considered under the assumption that the reaction coefficient depends nonlinearly on the substance's concentration. A diffusion coefficient is considered as the unknown coefficient and is recovered with the help of the additional information about the boundary value problem's solution. The inverse coefficient problem is reduced to multiplicative control problem, the solvability of which is proved in general form.

Key words nonlinear mass transfer model, generalized Oberbeck-Boussinesq model, inverse coefficient problem, multiplicative control problem

1. Introduction

For a long time an interest for the studying of inverse problems for linear and nonlinear models of heat-and-mass transfer doesn't fade. These mentioned problems consist in recovering of unknown densities of boundary or distributed sources of coefficients in differential equations of the model or in boundary conditions with the help of additional information about the system's state, which is described by a model. One of such methods of considering inverse problems is the optimisation method which implies the reduction of inverse or identification problems to extremum ones (see more in [1]).

The papers [2,3] are dedicated to the study of the inverse problems for a linear model of heat-and-mass transfer which consists of a convection-diffusion-reaction equation with boundary conditions. Further, let us also note the articles [4-12], in which inverse problems for nonlinear heat-and-mass transfer models in a classical approximation of Oberbeck-Boussinesq were considered. The papers [13-18] are focused on boundary value and extremum problems for convection-diffusion-reaction equation, in which reaction coefficient depends nonlinearly on substance's concentration. In [19,20] similar models of complex heat transfer were considered. From a range of papers dedicated to the study of boundary value and extremum problems for various models which generalize a classical approximation of Oberbeck-Boussinesq let us note [21-23]. About the research of more complicated rheological models and of models of multi-component viscous compressible fluids see, respectively, [24,25] and [26,27].

2. Boundary value problem

In a bounded domain $\Omega\subset\mathbf{R}^3$ with boundary Γ we consider the following boundary value problem:

$$-\nu\Delta\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\nabla \mathrm{p}=\mathbf{f}+\beta\mathbf{G}\varphi,\ \operatorname{div}\mathbf{u}=0 \text{ in } \Omega, \tag{2.1}$$

$$-\lambda\Delta\varphi+\boldsymbol{u}\cdot\nabla\varphi+k(\varphi,\boldsymbol{x})=f \text{ in } \Omega \tag{2.2}$$

$$\boldsymbol{u}=\mathbf{0},\ \ \varphi=0 \ on\ \Gamma. \tag{2.3}$$

Here $\mathbf{u}$ is a velocity vector, function φ presents the concentration of the pollutant, $p=P/\rho$, where P is pressure, ρ=const is the fluid density, ν=const>0 is the constant kinematic viscosity, $\lambda=\lambda(\mathbf{x})$ is the diffusion coefficient, β is the coefficient of mass expansion, $\mathbf{G}$=-(0,0,G) is the acceleration of gravity, $\mathbf{f}$ and f are volume densities of external forces or external sources of the substance, the function $k=k(\varphi,\mathbf{x})$ is the reaction coefficient, where $\mathbf{x}\in\Omega$. This problem (2.1)–(2.3) for given functions $\mathbf{f}$, f, λ, β and k will be called Problem 1 below.

In this paper the global solvability of Problem 1 is proved, sufficient condition for its solution's uniqueness are stated. Furthermore, with the help of the optimisation approach the problem of restoration of diffusion coefficient λ using the concentration φ, which was measured in a subdomain $Q\subset\Omega$, is reduced to the multiplicative control problem. Its solvability is proved in a general case. When the reaction coefficient and cost functionals are Frechet differentiable we get optimality systems for the extremum problem.

While studying the considered problems, we will use Sobolev functional spaces $H^s(D)$, $s\in\mathbf{R}$. Here D means either the domain Ω or some subset $Q\in\Omega$, or the boundary Γ. By $\|\cdot\|_{s,Q}$, $|\cdot|_{s,Q}$ and $(\cdot,\cdot)_{s,Q}$ we will denote the norm, seminorm and the scalar product in $H^s(Q)$. The norms and scalar products in $L^2(Q)$ and $L^2(\Omega)$ will be denoted correspondly by $\|\cdot\|_Q$ and $(\cdot,\cdot)_Q$, $\|\cdot\|_\Omega$ and $(\cdot,\cdot)_\Omega$.

Let

$$L^p_+(\Omega)=\{k\in L^p(\Omega): k\geq 0\}, p\geq \tfrac{3}{2},\ H^{3/2}_{\lambda_0}(\Omega)=\{h\in H^3(\Omega): h\geq\lambda_0>0\},$$

by

$$V=\{\boldsymbol{v}\in H^1_0(\Omega)^3: div\,\boldsymbol{v}=0\ in\ \Omega\}$$

we introduce the main functional space for velocity vector.

Let the following conditions hold:

(i) Ω is a bounded domain in the space $\mathbf{R}^3$ with boundary $\Gamma\in C^{0,1}$;

(ii) $\boldsymbol{f}\in L^2(\Omega)^3, f\in L^2(\Omega), \boldsymbol{b}=\beta\boldsymbol{G}\in L^2(\Omega)^3, \lambda\in H^{3/2}_{\lambda_0}(\Omega)$;

(iii) for any function $w\in H^1(\Omega)$ the embedding $k(w,\cdot)\in L^p_+(\Omega)$ is true for some $p\geq 3/2$, which does not depend on w, and on any sphere $B_r=\{w\in H^1_0(\Omega): ||w||_{1,\Omega}\leq r\}$ of radius r the following inequality takes place:

$$||k(w_1,\cdot)-k(w_2,\cdot)||_{L^p(\Omega)}\leq L\,||w_1-w_2||_{L^4(\Omega)}\ \ \forall\, w_1,w_2\in B_r.$$

Here L is the constant which depends on r, but does not depend on $w_1,w_2\in B_r$.

Let us note that the condition (iii) describes an operator, acting from $H^1(\Omega)$ to $L^p(\Omega)$, where $p\geq 3/2$, which gives us an opportunity to take into consideration the dependence of the reaction coefficient on either the component of solution the $(\mathbf{u},\varphi,p)$ of Problem 1 or on the spatial variable $\mathbf{x}\in\Omega$.

For example,

$$k_1=\varphi^2\ \left(or\ k_1=\varphi^2|\varphi|\right)\ \text{in subdomain } Q\subset\Omega \text{ or } k_1=k_0(x)\ in\ \Omega\backslash Q,$$
$$\text{where } k_0(x)\in L^2_+(\Omega\backslash Q).$$

From a physical point of view, the coefficient k_1 corresponds to the situation, when the substance's decomposition rate is proportional to the square (or cube) of substance's concentration in a subdomain $Q\subset\Omega$ and outside Q, and the rate of the chemical reaction depends only on a spatial variable [15,17].

The following theorem holds

Theorem 1. *If the conditions (i)–(iii) hold then there exists a weak solution* $(u,\varphi,p)\in V\times H^1_0(\Omega)\times L^2_0(\Omega)$ *of Problem 1 and the following estimates hold*

$$||\varphi||_1\leq M_\varphi,\qquad ||\boldsymbol{u}||_1\leq M_{\boldsymbol{u}},\qquad ||p||\leq M_p,$$

where $M_\varphi, M_{\boldsymbol{u}}$ *and* M_p *are non-decreasing function on initial date of Problem 1.*

3. Multiplicative control problem

In the framework of the optimisation approach the problem of the restoration of diffusion coefficient using an additional information about the solution of the Problem 1 can be reduced to the multiplicative control problem (see for example [12,18]). To state the control problem let us divide the whole set of initial data of the Problem 1 into two groups: group of fixed data which includes functions f, $\mathbf{f}$, $\mathbf{b}$ and $k(\varphi)$, and group of controls consisting of function λ. Here we assume that it can be changed in some set K.

We set $X = H_0^1(\Omega)^3 \times H_0^1(\Omega) \times L_0^2(\Omega)$, $\boldsymbol{x} = (\boldsymbol{u}, \varphi, p) \in X$ and introduce an operator $F = (F_1, F_2) : X \times K \to X^*$ by formula

$$\langle F_1(\boldsymbol{x},\lambda),(\boldsymbol{v},h)\rangle = \nu(\nabla\boldsymbol{u},\nabla\boldsymbol{v}) + (\lambda\nabla\varphi,\nabla h) + \big((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\boldsymbol{v}\big) - (p, div\,\boldsymbol{v}) +$$
$$+(k(\varphi)\varphi,h) + (\boldsymbol{u}\cdot\nabla\varphi,h) - (\boldsymbol{f},\boldsymbol{v}) - (\boldsymbol{b}\varphi,\boldsymbol{v}) - (f,h),$$
$$\langle F_2(\boldsymbol{x},\lambda),r\rangle = -(div\,\boldsymbol{u},r).$$

The equation F(**x**,λ)=0 is the operator form of weak formulation of Problem 1.

Let I is a weakly lower semicontinuous functional. Consider the following control problem:

$$J(\mathbf{x},\lambda) \equiv \frac{\mu_0}{2} I(\varphi) + \frac{\mu_1}{2}\|\lambda\|_{3/2,\Omega}^2 \to \inf,\ F(\mathbf{x},\lambda) = 0,\ (\mathbf{x},\lambda) \in X \times K. \tag{3.1}$$

Here μ_0 and μ_1 are non-negative parameters which are used to control the relative importance of terms in (28). Denote by

$$Z_{ad} = \{(\mathbf{x},\lambda) \in X \times K : F(\mathbf{x},\lambda) = 0, J(\mathbf{x},\lambda) < \infty\}$$

The set of feasible pairs for the problem (3.1) and assume that the following conditions hold

(j) $K \subset H_{\lambda_0}^{3/2}(\Omega)$ is nonempty convex closed set,

(jj) $\mu_0 > 0, \mu_1 \geq 0$ and K is a bounded set or $\mu_0 > 0, \mu_1 > 0$ and the functional *I* is bounded below.

The following cost functional can be used in the capacity of the possible one:

$$I_1(\varphi) = \|\varphi - \varphi^d\|_Q^2,\ I_2(\varphi) = \|\varphi - \varphi^d\|_{1,Q}^2,$$

$$I_3(\mathbf{u}) = \|\mathbf{u} - \mathbf{u}^d\|_Q^2,\quad I_4(p) = \|p - p^d\|_Q^2, \tag{3.2}$$

where function $\varphi^d \in L^2(Q)$ denote some desired concentration field given in a subdomain $Q \subset \Omega$. Functions $\boldsymbol{u}^d$ and p^d have similar sense for the velocity field ore pressure.

The following theorem holds

Theorem 2. *Let, under assumptions (i)-(ii) and (j), (jj) $I : X \to R$ be a weakly lower semicontinuous functional and a set Z_{ad} is nonempty. Then the control problem (28) has at least one solution $(\mathbf{x},\lambda) \in X \times K$.*

4. Conclusion

In conclusion, we studied boundary value and control problems for the generalized Oberbeck-Boussinescq model. In future papers we will obtain the optimality systems for the control problems (28). Using obtained systems, we will derive local stability estimates of optimal solutions.

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