UDC 517.58, 517.96

Fox-Wright Function on the boundary of its convergence domain

Dmitrii Karp[a] and Elena Prilepkina[b]

[a]Ton Duc Thang University, Ho Chi Minh City, Vietnam

email: dmitriibkarp@tdtu.edu.vn

[b]Far Estern Federal University, Vladivostok, Russia

email: pril-elena@yandex.ru

Abstract

This paper summarizes some of our recent results regarding the behavior of Fox-Wight function in the neighborhood of the singular point on the boundary of the disk of convergence. The main result is that the expansion contains two components - singular and regular as is the case for the generalized hypergeometric function. Recursive formulas for the coefficients of each of the two component series are also found.

Keywords: Fox-Wright function, Fox's H function, generalized hypergeometric function, singular point

Fox-Wright function ${}_p\Psi_q(z)$ is a further extension of the generalized hypergeometric function attained by introducing arbitrary positive scaling factors into the gamma functions in the summand:

$${}_p\Psi_q(z)=\sum_{k=0}^{\infty}\frac{\Gamma(A_1k+a_1)\cdots\Gamma(A_pk+a_p)}{\Gamma(B_1k+b_1)\cdots\Gamma(B_qk+b_q)}\frac{z^k}{k!}.$$

The importance of the Fox-Wright function comes mostly from its role in fractional calculus, see (Gorenflo et.al, 1999), (Kilbas, 2005), (Kilbas et.al., 2006), (Mainardi, Pagnini, 2007). Other interesting applications also exist. In particular, (Miller, 1991) expressed a solution of a general trinomial equation in terms of ${}_1\Psi_1(z)$. See, also (Miller, Moskowitz,1995) for an application in information theory. It has been a recent surge of interest in the Fox-Wright function as witnessed by the articles (Chu, Wang, 2008), (Mehrez, 2018), (Mehrez, Sitnik, 2018). In particular, Mehrez (Mehrez, 2018) extended our approach to generalized hypergeometric functions to the Fox-Wright function and presented the Laplace and the generalized Stieltjes transform representations for some cases of ${}_p\Psi_q(z)$ invoking certain facts from our previous works. He further obtained some of the inequalities and monotonicity results for certain cases of ${}_p\Psi_q(z)$, which we established previously for the generalized hypergeometric functions. In another recent work (Mehrez, Sitnik, 2018) the authors obtained further inequalities for the Fox-Wright functions and their ratios.

If the sums of the scaling factors in the denominator and numerator are equal the series defining this function has a positive and finite radius of convergence which we call ρ (otherwise if the sum of scaling factors in the denominator is greater than that if the numerator the series converges for all finite complex values of the variable and defines en entire function). In this report we are only interested in this case. The Fox-Wright function is then holomorphic inside the disk of convergence. The problem of its analytic continuation was completely solved by Braaksma in 1964 who proved that the Mellin-Barnes contour integral representation furnishes the analytic continuation to the entire complex plane cut along the ray $[1/\rho,\infty)$. It follows that the only singularity of the Fox-Wright function on the circle of radius ρ is at the point $z=1/\rho$. The character of this singularity is the main topic of this research. Moreover, when $z=1/\rho$ the series may converge or diverge depending on the values of parameters. Hence, we are encountered with the problem of analytic continuation of ${}_p\Psi_q(1/\rho)$ as the function of parameters $A_1,\dots,A_p,a_1,\dots,a_p,B_1,\dots,B_q,b_1,\dots,b_q$. Both problems (the behavior n the neighborhood of the singular point and analytic continuation in parameters) have a long history. Particular cases of these problems can be traced back to Gauss who found the celebrated summation formula for ${}_2F_1(a,b;c;1)$ providing analytic continuation in a, b, c, and described the behavior of $z\to{}_2F_1(a,b;c;z)$ in the neighborhood of $z=1$. Similar problems for ${}_3F_2$ were partly solved by Thomae in 1870 (analytic continuation in parameters) and partly by Ramanujan around the second decade of the 20th century (asymptotic approximation as $z\to1$ in the logarithmic case) with further contributions by Evans and Stanton, Wimp and Bühring, see (Bühring, 1987) and references therein. For the general Gauss type hypergeometric function ${}_{p+1}F_p$ with $p\geq3$ these problems were first solved by (Nørlund, 1955) with further contributions by (Olsson, 1966), (Saigo and Srivastava, 1990), (Bühring, 1992) and (Bühring and Srivastava, 1998) and other authors.

Our main results can be summarized as follows. First, we express the Fox-Wright function as the generalized Stieltjes transform of the delta-neutral H function of Fox, studied by us in (Karp, Prilepkina, 2017). Under certain restrictions on parameters this representation takes the from

$${}_p\Psi_q(z) = \int_0^1 \frac{H(\rho u)du}{u(1-\rho uz)^{\sigma}}.$$

Next, according to Theorem 1 from (Karp, Prilepkina, 2017) the H function in the integrand can be expanded as follows

$$H(\rho u) = u^{\theta+1}(1-u)^{\mu-1}\sum_{n=0}^{\infty} V_n\,(1-u)^n,$$

where the parameter θ is arbitrary real number, while μ is expressed by the parameters of the original function ${}_p\Psi_q(z)$. It's precise definition as well as formulas for computing the coefficients V_n can be found in (Karp, Prilepkina, 2017). Substituting this expansion into the integral representation above and performing term-wise integration (to be justified elsewhere) we arrive at the formula:

$${}_p\Psi_q(z) = \sum_{n=0}^{\infty}\frac{V_n}{\mu+n}\left({}_2F_1(1,\sigma;\mu+1+n;\rho z)\right)$$

We can now use the representation of the hypergeometric function ${}_2F_1$ in the neighborhood of the singular point $z=1$:

$${}_2F_1(1,\sigma;\,\mu+1+n;\rho z) = (1-\rho z)^{\mu-\sigma}\sum_{m=0}^{\infty} G_{m,n}(1-\rho z)^{m+n} + \sum_{m=0}^{\infty} D_{m,n}(1-\rho z)^{m}$$

After substituting this expansion into the above expression for the Fox-Wright function ${}_p\Psi_q(z)$ we can rearrange the summations to obtain:

$${}_p\Psi_q(z) = (1-\rho z)^{\mu-\sigma}\sum_{m=0}^{\infty} R_m(1-\rho z)^{m} + \sum_{m=0}^{\infty} W_m(1-\rho z)^{m}$$

The coefficients in this expansion can be computed easily from $G_{m,n}$ and $D_{m,n}$. Explicit formulas will be presented during the talk and published elsewhere. Here we just mention that the form of the above expansion with one "singular" series in fractional powers of $(1-\rho z)$ and one "regular" series in integer powers repeats the structure of the corresponding expansion of the generalized hypergeometric function. This result might be not very surprising, but rigorous confirmation of this expected form is given, to the best of our knowledge, for the first time here. Second important aspect of the above expansion is that assuming $\mu - \sigma > 0$ and setting z=1/ρ we obtain ${}_p\Psi_q(1/\rho)=W_0$, where W_0 is written as a convergent series in term of coefficients V_n. Hence, we get a formula for analytic continuation of ${}_p\Psi_q(1/\rho)$ in parameters $A_1,...,A_p,a_1,..,a_p,B_1,...,B_q,b_1,...,b_q$. This formula can be viewed as far reaching generalization of the Gauss formula for ${}_2F_1(a,b;c;1)$. Further consequences of the expansion around the singular point and integral representation via generalized Stieltjes transform of the delta-neutral H function will presented during the talk.

References

Braaksma B.L.J. Asymptotic expansions and analytic continuation for a class of Barnes integrals//Composito Math. 1962-64. Volume 15, Issue 3. pp. 239-341.

Bühring W. The behavior at unit argument of the hypergeometric function ${}_3F_2$//SIAM J. Math. Anal. 1987.Volome 18, Issue 5. pp.1227-1234.

Bühring W. Generalized hypergeometric functions at unit argument//Proceedings of American Mathematical Society. 1992. Volume 114. Number 1. pp.145-153.

Bühring W. and Srivastava H.M. Analytic continuation of the generalized hypergeometric series near unit argument with emphasis on the zero-balanced series, pages 17-35 in: Approximation Theory and Applications, Hadronic Press, 1998.

Kilbas A.A. and Saigo M. H-transforms and applications. Analytical Methods and Special Functions. Volume 9. Chapman & Hall/CRC, 2004.

Chu W. and Wang X. Summation formulae on Fox-Wright Psi-functions//Integral Transforms and Special Functions. 2008. Volume 19. Issue 8. pp.545-561.

Gorenflo R., Luchko Y., Mainardi F. Analytical properties and applications of the Wright function//Fractional Calculus and Applied Analysis. 1999. Volume 2. No.4. pp.383-414.

Karp D. and Prilepkina E. Some new facts around the delta neutral H function of Fox// Computational Methods and Function Theory, June 2017, Volume 17, Issue 2, pp. 343–367.

Kilbas A.A. Fractional calculus of the generalized Wright function//Fractional Calculus and Applied Analysis. 2005. Volume 8. pp.114-126.

Kilbas A.A., Saigo M., Trujillo J.J. On the generalized Wright function//Fractional Calculus and Applied Analysis. 2002. Volume 5. pp.437-460.

Kilbas A.A., Srivastava H.M., Trujillo J.J. Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies 204, Elsevier, 2006.

Kilbas A.A., Saxena R.K., Saigo M. and Trujillo J.J., Generalized Wright function as the H-function, in A.A.Kilbas and S.V.Rogosin (Eds.), Analytic Methods of Analysis and Differential Equations, AMADE 2003, Camb. Sci. Publ., Cambridge, 2006, 117-131.

Mainardi F. and Pagnini G. The role of the Fox-Wright functions in fractional sub-diffusion of distributed order//Journal of Computational and Applied Mathematics. 2007. Volume 207. No.2, pp.245-257.

Mehrez K. New integral representations for the Fox-Wright functions and its applications//J. Math.Anal.Appl. 2018. Volume 468. pp.650-673.

Mehrez K. and Sitnik S.M. Functional Inequalities for Fox-Wright Functions//The Ramanujan Journal. 2018. DOI: 10.1007/s11139-018-0071-2.

Miller A.R. Solutions of Fermat's last equation in terms of Wright function//Fibonacci Quarterly. 1991. Volume 29. pp.52-56.

Miller A.R. and Moskowitz I.S. Reduction of a Class of Fox-Wright Psi Functions for Certain Rational Parameters//Computers Math. App. 1995. Volume 30. No. 11. pp. 73-82.

Nørlund N.E. Hypergeometric functions//Acta Mathematica, 1955. 94. pp.289-349.

Olsson P.O.M. Analytic continuation of higher-order hypergeometric functions//Journal of Mathematical Physics. 1966. Volume 7. no.4. pp.702-710.

Saigo M. and Srivastava H.M. The behaviour of zero balanced hypergeometric series ${}_pF_{p-1}$ near the boundary of its convergence region//Proceedings of the American Mathematical Society. 1990. Volume 110. Issue 1. pp.71-76.