UDC 512.53, 512.58

UPPER SEMILATTICES S FOR WHICH THE CONGRUENCE LATTICE OF S-ACT $_SS$ IS MODULAR.

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Abstract. The works of such authors as Ptakhov D.O., Stepanova A.A., Haliullina A.R., Kazak M.S. are devoted to studying of the S-acts over monoids with the given conditions on their congruence lattices. It is clear, that upper semilattice with the smallest element can be considered as monoid. Stepanova A.A. and Kazak M.S. described S-acts over upper semilattice with linear congruence latticeю More exactly they prove that S-act A over upper semilattice S has linear ordered congruence lattice if and only if A has at most 2 elements. In this research we study the structure of upper semilattices S for which the congruence lattice of S-act S is modular.
Key words: lattice, modular lattice, congruence lattice of algebra, S-act.

The subject of our research in this article are S-acts over monoids, that is the sets on which monoids act. S-acts can be found in different sections of mathematics and its applications. In particularly, every unary algebra is S-act over monoid of its endomorphisms. The same holds true for a graph, partially ordered set, etc. The main ideas of the theory of S-acts were reflected in the monograph [6].

The congruences of algebra play an important role in a structural theory since the congruences are the same as kernel of the homomorphism of this algebra into the others. The congruences of algebra forms the lattice according to set-theoretic inclusion. This lattice is sublattice of the lattice of all equivalence relations. The lattice carries much information about the structure of algebra.

Many works have been devoted to studying of the S-acts over monoids with the given conditions on their congruence lattices. From the definition of S-act it follows that S-act is a unary algebra. Studies related to the description of S-acts with some conditions on their congruence lattice, began with the study of unary algebras with linear ordered, distributive and modular congruence lattice. The congruence lattice of unary algebras, in particularly S-acts over monoids, has been studied in the works [1-4,6,7].

In [1] Egorova D.P. described the structure of the unary with linear, distributive and modular congruence lattice.

In [7] Ptakhov D.O. and Stepanova A.A. studied the unconnected S-acts with distributive and modular congruence lattice. In particularly, in this paper it is noticed that S-act with a distributive congruence lattice has at most three connected components. In addition, it is noticed that S-act with a modular congruence lattice has at most four connected components.

In [2] Haliullina A.R. got the characterization of S-acts over semigroups S of right and left zeros with distributive lattice of congruence.

In [4] Kazak M.S. and Stepanova A.A. described the S-acts over a well-ordered monoid such that their congruence lattice is distributive, or modular, or linear ordered. In the theorems bellow $(S; \leq)$ is an linear ordered set with the smallest element 1 considered as a (linear ordered) monoid $(S; \cdot)$ relative to the operation $ab = \max\{a, b\}$, where $a, b \in S$.

Theorem 1. [4] Let S be a linear ordered monoid. Then S-act A has linear ordered congruence lattice if and only if A has at most 2 elements.

Theorem 2. [4] Let S be a well-ordered monoid. Then S-act A has distributive congruence lattice if and only if

1) S-act A has at most two connected components;

2) if a, b ∈ A, s ∈ S, s is not 1 and $Sa_1 \cap Sa_2 = Ssa_1 \cap Ssa_2$ then $sa_1=ra_1$ or $sa_2=ra_2$ for some s ∈ S, r<s.

Theorem 3. [4] Let S be a well-ordered monoid. Then S-act A has modular congruence lattice if and only if

1) S-act A has at most three connected components;

2) if a, b ∈ A, s ∈ S, s is not 1 and $Sa_1 \cap Sa_2 = Ssa_1 \cap Ssa_2$ then $sa_1=ra_1$ or $sa_2=ra_2$ or $sa_1=sa_2$ for some s ∈ S, r<s;

3) if a_1, a_2,a_3 ∈ A and s ∈ S such that s is not 1, S-act Sai is not subact of S-act Sa_j, $sa_i=sa_j$ and $Sa_i \cap Sa_j = Ssa_1$ for any different i,j∈{1,2,3} then $sa_i=ra_i$ for some s ∈ S, r<s and i,j∈{1,2,3}.

In [5] Kazak M.S. and Stepanova A.A. studied the structure S-act with linear ordered congruence lattice where S is an upper semilattice with the smallest element 1. In the theorem bellow (S; ≤) is an upper semilattice with the smallest element 1 considered as a monoid (S; ·) relative to the operation ab = max{a, b}, where a, b ∈ S.

Theorem 4. [5] Let S be an upper semilattice. Then S-act A has linear ordered congruence lattice if and only if A has at most 2 elements.

In this work we study the structure of upper semilattices S for which the congruence lattice of S-act S is modular.

Below we recall some definitions from the theory of S-acts and universal algebra [6,8].

A subalgebra θ of a direct product of algebra A is called a *congruence* if
1) $(a,a) \in \theta$ for all $a \in A$;
2) if $(a,b) \in \theta$ then $(b,a) \in \theta$ for all $a,b \in A$;
3) if $(a,b) \in \theta$ and $(b,c) \in \theta$ then $(a,c) \in \theta$ for all $a,b,c \in A$.
The set of all congruences of algebra A forms the lattice according to following operations:

$$\theta_1 \wedge \theta_2 = \theta_1 \cap \theta_2,$$

$\theta_1 \vee \theta_2$ is the smallest congruence included $\theta_1 \cup \theta_2$,
where θ_1, θ_2 are the congruences of algebra A.

A partially ordered set $(L; \leq)$ is called *upper semilattice* (*lattice*) if every 2-element subset $\{a,b\} \subseteq L$ has the least upper bound bond $a \vee b$ (and the greatest lower $a \wedge b$). A lattice $(L; \leq)$ is called *modular lattice* if

$$(a \vee b) \wedge c = a \vee (b \wedge c)$$

for all $a,b,c \in L$ such as $a \leq c$.

A *monoid* is a semigroup with identity 1. Everywhere below the S will denote a monoid with identity 1. *Left S-act* is a non-empty set A on which S acts in the left, that is there is a map $S \times A \to A$, where $(s,a) \mapsto sa$, such that for all $s,t \in S \; and \; a \in A$,
$s(t(a)) = (st)a$ and $1\,a = a$.
Note that monoid S is S-act over S.

Suppose (S; ≤) be an upper semilattice with the smallest element 1 considered as a monoid (S; ·) relative to the operation *ab* = $a \vee b$ where $a,b \in S$.

Theorem 5. Let S be upper semilattice with element 1. A lattice of congruence of S-act S is modular if and only if

$$\neg((b<a) \wedge (b<c))$$

for all different $a,b,c \in S$.

The reported study was funded by RFBR according to the research project №17-01-0053.

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