UDC 517.58, 517.96

On the generalized hypergeometric functions with integral parameter differences

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Abstract

Miller-Paris transformations are extensions of Euler's transformation for the Gauss hypergeometric functions to higher-order generalized hypergeometric functions with integral parameter differences (IPD). Some generalizations of Miller-Paris transformations were prooved by us to obtain new results for the generalized hypergeometric function with IPD. In particular, we give several extensions of the Karlsson-Minton summation formula.

Keywords: generalized hypergeometric function, Karlsson-Minton summation, Miller-Paris transformation, hypergeometric summation, integral parameter differences

Suppose **a** and **b** are complex vectors, $(a)_n$ is the Pochhammer's symbol. We will use the following convenient abbreviated notation:

$$\Gamma(\mathbf{a})=\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_p),\quad (\mathbf{a})_n=(a_1)_n(a_2)_n\cdots(a_p)_n,\quad \mathbf{a}+\mu=(a_1+\mu,a_2+\mu,\ldots,a_p+\mu)$$

Inequalities like Re(**a**)>0 will be understood element-wise.

The generalized hypergeometric functions is defined by the series

$${}_pF_q\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle|\,x\right)=\sum_{n=0}^{\infty}\frac{(\mathbf{a})_nx^n}{(\mathbf{b})_nn!}\qquad(1)$$

for the values of x making the series convergent and by analytic continuation elsewhere. The size of a vector appearing in the parameters of the generalized hypergeometric function can usually be read off its indices. More detailed information about the definition and properties of the generalized hypergeometric function can be found in standard textbooks (Andrews, Askey and Roy, 1999), **(**Beals and Wong, 2016), (Luke, 1969) and reference books (Olver., Lozier, Boisvert and Clark (Eds.), 2010). The argument x=1 of the generalized hypergeometric function will be routinely omitted. In what follows we also will use the notation

$$\mathbf{m}=(m_1,\ldots,m_r)\in\mathbb{N}^r,\ m=m_1+m_2+\ldots+m_r,\ \mathbf{f}=(f_1,\ldots,f_r),\ (\mathbf{f})_{\mathbf{m}}=(f_1)_{m_1}\cdots(f_r)_{m_r},$$
$$(\mathbf{f}-b)_{\mathbf{m}}=(f_1-b)_{m_1}\cdots(f_r-b)_{m_r}.$$

In 1970 Minton (Minton, 1970) proved the summation formula

$${}_{r+2}F_{r+1}\left(\begin{matrix}-k,b,\mathbf{f}+\mathbf{m}\\ b+1,\mathbf{f}\end{matrix}\right)=\frac{k!}{(b+1)_k}\frac{(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}}\qquad(2)$$

valid for $k\geq m$, k is a natural number. Soon thereafter, his result was generalized by Karlsson (Karlson, 1971) who replaced $-k$ by an arbitrary complex number a satisfying Re(1-a-m)>0 to get

$$ {}_{r+2}F_{r+1}\left(\begin{matrix} a, b, \mathbf{f}+\mathbf{m} \\ b+1, \mathbf{f} \end{matrix}\right) = \frac{\Gamma(b+1)\Gamma(1-a)}{\Gamma(b+1-a)} \frac{(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}}. \tag{3} $$

Gasper (Gasper, 1981) deduced a q-analogue and a generalization of Minton's and Karlsson's formulas; Chu (Chu, 1994), (Chu, 1995) found extensions to bilateral hypergeometric and q-hypergeometric series; their results were re-derived by simpler means and further generalized by Schlosser (Schlosser, 2001), who also found multidimensional extensions to hypergeometric functions associated with root systems (Schlosser, 2003). For further developments in this directions, see also (Rosengren, 2004) and (Rosengren, 2005). Quite recently, Miller and Srivastava (Miller and Srivastava, 2010) found some alternative derivations of the original Karlsson and Minton formulas. Some of the results mentioned above found applications in the theory of rook polynomials and related combinatorial problems as discovered by Haglund in (Haglund, 1996), who also noticed that some of the results can be obtained by specializing a very general transformation due to Slater. We also mention an interesting work (Letessier, Valent and Wimp, 1994), where the reduction of the differential equation satisfied by the generalized hypergeometric functions with some integer parameter differences was found.

Conditions $k \geq m$ in (2) and $\mathrm{Re}(1-a-m)>0$ in (3) can not be dispensed with: the formulas become false when they are violated (take for example $a=-4$, $b=33/17$, $\mathbf{f}=(4.2;-5/3)$, $\mathbf{m}=(7;8)$). The situation with (3) is thereby quite subtle: the function on the right hand side is continuous at $a=1-m$ while the function on the left hand side is finite but has a discontinuity at this point. Further, if $\mathrm{Re}(1-a-m)<0$ and a is not a negative integer or zero, the series on the left hand side diverges. At the same time, it reduces to a finite sum for $a=-k$, so that such points are "isolated summability points" and analytic continuation of the right hand side in a does not lead to the correct value of the left hand side at such point.

In this paper we discuss the correct value of the left hand side of Minton's formula (2) at these special points $k=0,1,...,m-1$. In particular, we have obtained the following results

$$ {}_{r+2}F_{r+1}\left(\begin{matrix} 1-m, b, \mathbf{f}+\mathbf{m} \\ b+1, \mathbf{f} \end{matrix}\right) = \frac{(m-1)!}{(b+1)_{m-1}} \frac{(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}} - \frac{(-1)^m (m-1)! b}{(\mathbf{f})_{\mathbf{m}}} $$

and

$$ {}_{r+2}F_{r+1}\left(\begin{matrix} 2-m, b, \mathbf{f}+\mathbf{m} \\ b+1, \mathbf{f} \end{matrix}\right) = \frac{(m-2)!}{(b+1)_{m-2}} \frac{(\mathbf{f}-b)_{\mathbf{m}}}{(\mathbf{f})_{\mathbf{m}}} $$
$$ + \frac{(-1)^m (m-2)! b}{(\mathbf{f})_{\mathbf{m}}} \left(1 - b + m(m-2+f_r) + \sum_{q=1}^{r-1} (f_q - f_{q+1} - m_{q+1}) \sum_{i=1}^{q} m_i \right). $$

In addition we present an extension of Karlsson's formula (3) obtained by replacing the number b by a vector $\mathbf{b}$ and the number $b+1$ by the vector $\mathbf{b}+\mathbf{p}$, where $\mathbf{p}$ is a positive integer vector. Suppose $\mathbf{b}=(b_1,...,b_l)$ is a complex vector, $\mathbf{p}=(p_1,...,p_l)$ is a vector of positive integers, $p=p_1+...+p_l$, and all elements of the vector

$$ \boldsymbol{\beta} = (b_1, b_1+1, \ldots, b_1+p_1-1, \ldots, b_l, b_l+1, \ldots, b_l+p_l-1) = (\beta_1, \beta_2, \ldots, \beta_p) $$

are distinct. If $\mathrm{Re}(p-a-m)>0$, then

$$ \frac{1}{\Gamma(1-a)} {}_{r+l+1}F_{r+l}\left(\begin{matrix} a, \mathbf{b}, \mathbf{f}+\mathbf{m} \\ \mathbf{b}+\mathbf{p}, \mathbf{f} \end{matrix}\right) = \frac{(\mathbf{b})_{\mathbf{p}}}{(\mathbf{f})_{\mathbf{m}}} \sum_{q=1}^{p} \frac{\Gamma(\beta_q)(\mathbf{f}-\beta_q)_{\mathbf{m}}}{B_q \Gamma(1+\beta_q-a)}, $$

where

$$ B_q = \prod_{v=1,\, v\neq q}^{p} (\beta_v - \beta_q). $$

In particular,

$$\frac{1}{\Gamma(1-a)}{}_{r+2}F_{r+1}\left(\begin{matrix}a,b,\mathbf{f}+\mathbf{m}\\ b+p,\mathbf{f}\end{matrix}\right)=\frac{\Gamma(b+p)}{(\mathbf{f})_{\mathbf{m}}}\sum_{q=0}^{p-1}\frac{(b)_q(\mathbf{f}-b-q)_{\mathbf{m}}}{(p-q-1)!(-q)_q\Gamma(1+q+b-a)}.$$

Further, we apply some generalizations Miller-Paris transformations to obtain new formula for the generalized hypergeometric function with integral parameters differences in the unit. To formulate the result we will need the polynomials

$$Q_m(b,c,\mathbf{f},\mathbf{m};t)=\frac{(c-b-t-m)_m}{(c-b-m)_m}\sum_{k=0}^{m}{}_{r+1}F_r\left(\begin{matrix}-k,\mathbf{f}+\mathbf{m}\\ \mathbf{f}\end{matrix}\right)\frac{(t)_k(b)_k}{(1+t+b-c)_k k!}.$$

and

$$\hat{Q}_m(a,b,c,\mathbf{f},\mathbf{m};t)=\sum_{k=0}^{m}\frac{(-1)^k C_{k,r}(a)_k(b)_k(t)_k}{(c-a-m)_k(c-b-m)_k}{}_3F_2\left(\begin{matrix}-m+k,t+k,c-a-b-m\\ c-a-m+k,c-b-m+k\end{matrix}\right),$$

where

$$C_{k,r}=\frac{(-1)^k}{k!}{}_{r+1}F_r\left(\begin{matrix}-k,\mathbf{f}+\mathbf{m}\\ \mathbf{f}\end{matrix}\right).$$

Suppose $\mathbf{h}=(h_1,\ldots,h_l)$ is a complex vector and $\mathrm{Re}(c+e-a-b-d-p-m)>0$, $\mathrm{Re}(e-d-p)>0$. Then

$${}_{r+l+3}F_{r+l+2}\left(\begin{matrix}a,b,d,\mathbf{h}+\mathbf{p},\mathbf{f}+\mathbf{m}\\ c,e,\mathbf{h},\mathbf{f}\end{matrix}\right)=\mathcal{A}_{m+p+3}F_{m+p+2}\left(\begin{matrix}c-a-m,c-b-m,d,\boldsymbol{\nu}+1,\boldsymbol{\eta}+1\\ c,e+\kappa,\boldsymbol{\nu},\boldsymbol{\eta}\end{matrix}\right)$$

where

$$\mathcal{A}=\frac{(\boldsymbol{\zeta}+\kappa)_1\Gamma(e)\Gamma(e-d-p+\kappa)}{(\boldsymbol{\zeta})_1\Gamma(e+\kappa)\Gamma(e-d-p)}.$$

$\kappa=c-a-b-m$ are the roots of the polynomial $Qp(d,e,\mathbf{h},\mathbf{p};t)$, $\boldsymbol{\nu}$ are the roots of the polynomial

$$Q_p(e-d+\kappa-p,e+\kappa,\boldsymbol{\zeta}+\kappa,\mathbf{1};t)$$

$\boldsymbol{\eta}$ are the roots of the polynomial

$$\hat{Q}_m(a,b,c,\mathbf{f},\mathbf{m};t).$$

The presented results are partially included in the publications (Karp and Prilepkina, 2018) and (Karp and Prilepkina, 2019).

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