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PRIMITIVE NORMALITY OF THE CLASS OF PRINCIPALLY WEAKLY INJECTIVE S-ACTS

Alena Stepanova, Evgenii Efremov
Far Eastern Federal University, Russia
stepltd@mail.ru
efremov-el@mail.ru

Abstract. One of the standard problems in the model theory of S-acts is to describe the monoids over which a certain class of S-acts has some model-theoretic properties. Primitive normality of the class of all S-acts and the class of all injective S-acts was studied in works of Stepanova A.A. and Efremov E.L. Namely, it is proved that the class of all injective S-acts is primitive normal. The notion of principally weakly injective S-act can be considered as a natural generalization of the notion of injective S-act. In this article we describe monoids S such that the class of principally weakly injective S-acts is primitive normal. It is shown that the class of principally weakly injective S-acts is primitive normal if and only if S is linearly ordered monoid.

Keywords: theory, primitive normal theory, monoid, S-act, principally weakly injective S-act.

This theme relates to model-theoretic algebra, that is a branch of mathematics that connects the model theory with other areas of mathematics. The subject of the research is the class of principally weakly injective S-acts.

The model theory of modules over a ring has long been a respectable branch of both model theory and ring theory. The concept of an act over a monoid S (S-act) is a generalization of the concept of a module over a ring. The model theory of S-acts is rather less developed but again exhibits a nice interplay between algebra and model theory, with its own distinct flavor. Many problems in model theory of S-acts came from the module model theory. For example, the problem of an algebraic description of monoids S over which some classes of S-acts have some model-theoretic properties (axiomatizability, completeness, stability, primitive normality, primitive connectivity, etc.). These properties for classes of projective, flat, regular, free S-acts were studied in the works of Gould V., Poizat B., Mustafin T.G., Stepanova A.A., Ovchinnikova E.V. and others.

The characterization of the monoids S with axiomatizable and model complete class of regular S-acts was given in [8]. In this article the authors describe the monoids with complete class of regular S-acts which satisfy the additional conditions. They studied the monoids S, all regular S-acts over which have the stable and superstable theory. The authors proved the stability of the axiomatizable model complete class of regular S-acts. They also described the monoids S with the superstable and ω-stable class of regular S-acts when this class is axiomatizable and model complete.

In [10] the authors investigate the commutative monoids over which the axiomatizable class of regular S-acts is primitive normal and antiadditive. They proved that the primitive normality of an axiomatizable class of regular S-acts over the commutative monoid S is equivalent to the antiadditivity of this class and it is equivalent to the linearity of the order of a semigroup R such that an S-act ${}_SR$ is a maximal (under the inclusion) regular subact of the S-act ${}_SS$.

The monoids S over which the class of all regular S-acts is axiomatizable and primitive connected were studied in [13]. Stepanova A.A. proved that the axiomatizable class of all regular S-acts is primitive connected if and only if the semigroup R is rectangular band of groups and $R = eR$ for some idempotent $e \in R$, where ${}_SR$ is the inclusion maximal regular S-subact in the S-act ${}_SS$.

Questions of axiomatizability, completeness, model completeness and stability for classes of free, projective and flat S-acts were considered in [5].

The difficulty of studying the model-theoretical properties of classes of injective S-acts, in contrast to projective, flat, free S-acts, is the absence of an algebraic description of injective S-acts. The first work in this direction can be considered the work of Stepanova A.A. [9], where she described commutative monoids over which the class of all injective S-acts is axiomatizable, complete and model complete. The continuation of these studies is the work of Stepanova A.A. and Efremov E.L. where the structure of monoids S with axiomatizable classes of weakly injective S-acts, finitely generated weakly injective S-acts and principally weakly injective S-acts has been described [3].

Complete, model-complete, stable and superstable classes of injective and weakly injective S-acts were studied in [1]. It is shown that for a right-reversible monoid S the class of injective (weakly injective, finitely generated, weakly injective) S-acts is complete only if S is trivial. For an arbitrary monoid S it is shown that the completeness of the class of principally weakly injective S-acts is equivalent to the triviality of the monoid S. For a finite monoid S it is proved that the stability (superstability) of the theory of any injective, weakly injective, finitely generated weakly injective S-act is equivalent to the fact that S is a linearly ordered monoid (fully ordered monoid). In addition, it is shown that for an arbitrary monoid S a class of principally weakly injective S-acts is stable (superstable) if and only if S is a linearly ordered monoid (fully ordered monoid).

In this article, we describe the monoids S such that the class of principally weakly injective S-acts is primitive normal.

Below we recall some facts of the theory of S-acts, model theory and universal algebra [4, 6-7].

Throughout this article, S is a monoid with neutral element 1.

An algebraic system $\langle A; s\rangle_{s\in S}$ of signature $L_S = \{s \mid s \in S\}$ is called *a (left) S-act*, or *an act over S*, or simply *an S-act*, whenever $s_1(s_2a) = (s_1s_2)a$ and $1a = a$ for all $s_1, s_2 \in S,\ a \in A$. We denote an S-act $\langle A; s\rangle_{s\in S}$ by ${}_SA$. All S-acts in this article are left S-acts. Each subsystem ${}_SB$ of an S-act ${}_SA$ is called *a S-subact* of ${}_SA$ and denoted by ${}_SB \subseteq {}_SA$.

We say that a homomorphism $\varphi\colon {}_SB \to {}_SC$ of S-acts *extends* a homomorphism $f\colon {}_SA \to {}_SC$ of S-acts, where ${}_SA \subseteq {}_SB$ whenever $\varphi\mid_A = f$, i.e. $\varphi(a) = f(a)$ for all $a \in A$.

An injective S-act is an S-act ${}_SQ$ such that for any S-act ${}_SA$, for any S-subact ${}_SB$ of ${}_SA$ and for any homomorphism $\varphi\colon {}_SA \to {}_SQ$ there exists a homomorphism $\bar{\varphi}\colon {}_SB \to {}_SQ$ which extends φ, i.e., $\bar{\varphi}$ is such that the diagram

$$\begin{tikzcd} {}_SA \arrow[d, "\varphi"'] \arrow[r, phantom, "\subseteq"] & {}_SB \arrow[dl, "\bar{\varphi}"] \\ {}_SQ & \end{tikzcd}$$

is commutative.

A principally weakly injective S-act is an S-act ${}_SQ$ such that for any principal left ideal Ss of S and for any homomorphism $\varphi\colon {}_SSs \to {}_SQ$ there exists a homomorphism $\bar{\varphi}\colon {}_SS \to {}_SQ$ which extends φ.

We denote by **S-PWInj** the class of all principally weakly injective S-acts.

A monoid S is called *a linear ordered monoid* if the set $\{Ss \mid s \in S\}$ is linear ordered with respect to inclusion.

Let T be a complete theory of signature L, $\mathcal{A} = \langle A; L\rangle$ be an algebraic system.

A tuple $\langle a_1, \dots, a_n\rangle$ of elements of A and a tuple $\langle x_1, \dots, x_n\rangle$ of variables are denoted by $\bar{a}$ and $\bar{x}$, respectively. Let $\bar{v}$ be tuple of elements or variables. We introduce the following notation: $l(\bar{v})$ is the length of $\bar{v}$, $\bar{v}(i)$ is the i-th element of $\bar{v}$. Instead of $\bar{a} \in A^n$ we write $\bar{a} \in A$.

If $\Phi(\bar{x}, \bar{y})$ is a formula of signature L, $\bar{a} \in A$, $l(\bar{a}) = l(\bar{y})$, then we denote by $\Phi(\mathcal{A}, \bar{a})$ the set $\{\bar{b} \in A \mid \mathcal{A} \vDash \Phi(\bar{b}, \bar{a})\}$.

A formula $\exists\bar{x}(\Phi_0 \wedge \dots \wedge \Phi_k)$, where Φ_i $(i \le k)$ are atomic formulas of signature L, is said to be *primitive*. Let $\Phi(\bar{x}, \bar{y})$ be a primitive formula of signature L, $\bar{a} \in A$, $l(\bar{a}) = l(\bar{y})$. Then we say that a set $\Phi(\mathcal{A}, \bar{a})$ is *primitive*. If $\bar{b} \in A$ and $l(\bar{b}) = l(\bar{y})$, then the sets $\Phi(\mathcal{A}, \bar{a})$ and $\Phi(\mathcal{A}, \bar{b})$ are called *primitive copies*.

A theory T is said to be *primitive normal* if $X = Y$ or $X \cap Y = \emptyset$ for any primitive copies X and Y. A class $\mathcal{K}$ of algebraic systems of signature L is called *primitive normal* if a theory $\mathrm{Th}(\mathcal{A})$ is primitive normal for any algebraic system $\mathcal{A} \in \mathcal{K}$.

Theorem. *The class* **S-PWInj** *of all principally weakly injective S-acts is primitive normal if and only if S is a linearly ordered monoid.*

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