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MIRIMANOFF POLYNOMIALS: APPROACH TO HYPERGEOMETRIC AND GENERATING FUNCTIONS

Abstract. The aim of the this presentation is to survey and study some fundamental properties of the Mirimanoff polynomials with the aid of hypergeometric functions (hypergeometric series) and Apostol type and Frobenius type polynomials and numbers. We give some formulas and identities for these polynomials. The results of this presentation will be discussed comparatively with the results given previously studies and works.
Key words Mirimanoff polynomials, Frobenius-Euler numbers and polynomials, Bernoulli and Euler numbers and polynomials, Generating functions, Hypergeometric functions.

Introduction

Hypergeometric series and generating functions are among the most widely used subjects of physics, engineering and other sciences, especially in almost all fields of mathematics. Using the techniques and methods of these functions, the results of this presentation will be presented.

The Mirimanoff polynomials are defined by the following formula:

$$f_m(x,n)=\sum_{j=0}^{n-1} j^m x^j \tag{1}$$

(*cf*. Carlitz 1959; Vandiver 1942).
Equation (1) is also modified as follows:

$$f_m(x,n,k)=\sum_{j=0}^{n-1} (j+k)^m x^j \tag{2}$$

(*cf*. Carlitz 1959; Vandiver 1942). By using (2), we have

$$f_m(x,n,k)=\sum_{v=0}^{m} \binom{m}{v} k^{m-v} f_v(x,n).$$

In [Simsek 2019], we gave

$$\left(\frac{1}{\lambda}\right)^{1-n} f_m\left(\frac{1}{\lambda},n,x\right)=\frac{1}{1-\lambda}\left(H_m(x+n;\lambda)-\lambda^n H_m(x;\lambda)\right)$$

where $H_m(x;\lambda)$ denotes the Frobenius-Euler polynomials which are defined by means of the following generating function:

$$G_F(t,x;u)=\frac{1-u}{e^t-u}e^{xt}=\sum_{n=0}^{\infty}H_n(x;u)\frac{t^n}{n!} \tag{3}$$

(*cf.* Apostol 1951-Vandiver 1942).
Substituting $u=-1$ into (3), we get

$$H_n(x;-1)=E_n(x)$$

where $\mathrm{E_n}(x)$ denotes the Euler polynomials.
The polynomials $P(x;m,n;\lambda,p)$ including sums of higher power of binomial coefficients are defined by means of the following generating function:

$$G(t,x;\lambda,n,p)=\frac{1}{n!}\ {}_pF_{p-1}\left[\begin{matrix}-n,-n,\dots,-n\\1,1,\dots,1\end{matrix};(-1)^p\lambda e^t\right]\ {}_pF_p\left[\begin{matrix}1,1,\dots,1\\1,1,\dots,1\end{matrix};xt\right]$$

$$=\sum_{m=0}^{\infty}P(x;m,n;\lambda,p)\frac{t^m}{m!}$$

where $n,p\in\mathbb{N}$ and $\lambda\in\mathbb{R}$ or $\mathbb{C}$ (*cf.* Djordjevic, Milovanovic 2014; Koepf 2014; Temme 1996).
In [Simsek 2019], we gave

$$f_m\left(\frac{1}{\lambda},n,x\right)=n!\,P\left(x;m,n-1;\frac{1}{\lambda},0\right).$$

Substituting $x = a$ into (1), we have

$$f_m(a,n)=\sum_{j=0}^{n-1}j^m a^j=\frac{a^{m+1}\mathfrak{B}_{m+1}(n;a)-\mathfrak{B}_{m+1}(a)}{m+1}$$

where $\mathfrak{B}_m(x;\lambda)$ denotes the Apostol Bernoulli polynomials which are defined by means of the following generating function:

$$\frac{t}{\lambda e^t-1}e^{xt}=\sum_{n=0}^{\infty}\mathfrak{B}_n(x;\lambda)\frac{t^n}{n!}$$

where $\mathfrak{B}_n(\lambda)=\mathfrak{B}_n(0;\lambda)$denotes the Apostol Bernoulli numbers (*cf.* Apostol 1951-Vandiver 1942).
Substituting $a=1$ into the above equation, we have

$$f_m(1,n)=\sum_{j=0}^{n-1}j^m=\frac{B_{m+1}(n)-B_{m+1}}{m+1},$$

where $B_{m+1}(n)$ denotes Bernoulli polynomials (*cf.* Apostol 1951-Vandiver 1942).
Substituting $x = -1$ into (1), we have

$$f_m(-1,n)=\sum_{j=0}^{n-1}(-1)^j j^m=\frac{(-1)^{n-1}E_{m+1}(n)-E_{m+1}}{2}$$

(*cf.* Kim 2005-Simsek 2019) where $E_m(x)$ denotes the Euler polynomials which are defined by means of the following generating function:

$$\frac{2}{e^t+1}e^{xt}=\sum_{m=0}^{\infty}E_m(x)\frac{t^m}{m!}$$

where E_m=$E_m(0)$ denotes the Euler numbers (*cf*. Apostol 1951-Vandiver 1942).

Conclusion

The result of this paper is related to the polynomials $P(x;m,n;\lambda,p)$ which related to the various kind of the special numbers and polynomials and also special functions including the Mirimanoff polynomials, the Frobenius-Euler numbers and polynomials, the Bernoulli numbers and polynomials, the Euler numbers and polynomials. The results of this presentation may give potentials applications for other areas.

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